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OPTIMAL ALLOCATION OF TACTICAL MISSILES BETWEEN VALUED TARGETS AND DEFENSE TARGETS

by

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Various combat duels between an attacker, who owns a stockpile of long range precision-guided missiles, and a defender are addressed. The defender must defend a valued target, or several selected targets (called primary targets) by a group of defending targets (called secondary targets, and are usually understood to be surface-to-air missile batteries). The problem of the attacker is to allocate his missiles between the primary and the secondary

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(a) Models in which the attacker must find optimal sequencing of missiles which are either anti-primary or anti-secondary missiles.

(b) Models in which the attacker must find optimal sequencing of missiles which are either real (anti-primary) missiles or decoys. Two mechanisms by which decoys may enhance effectiveness, namely, exhaustion and saturation of the defense, are quantitatively explored.

Various cases are examined in the thesis, which makes a heavy use of stochastic dynamic programming and sequential games techniques. Some numerical examples are also given.

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Optimal Allocation of Tactical Missiles
Between Valued Targets and Defense Targets

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LIST OF ABBREVIATIONS

ABM	-	Anti-Ballistic Missile
RV	-	Reentry Vehicle
A/G	-	Air-to-Ground
G/G	-	Ground-to-Ground
G/A	-	Ground-to-Air
SAM	-	Surface-to-Air Missiles
AAA	-	Anti-aircraft Artillery
MOE	-	Measure of Effectiveness
CEP	-	Circular Probable Error
EW	-	Electronic Warfare
ECM	-	Electronic Warfare Countermeasures
AR	-	Anti-Radiation
MPH	-	Maximum Probability of Hit (Criterion)
MENP	-	Maximum Expected Number of Penetrators (Criterion)
MEC	-	Minimum Expected Cost (Criterion)
ASAPA	-	Anti-Secondary/Anti-Primary Allocation (Game Model)
AP	-	Anti-Primary
AS	-	Anti-Secondary
MPR	-	Miss Probability Ratio
MMPR	-	Monotone Miss Probability Ratio

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I. INTRODUCTION

A. MOTIVATION AND CONCEPTUAL BACKGROUND

This work contains an analysis of various tactical decision processes involving the use of long range conventional tactical missiles. It is written with a view towards the future of tactical missiles warfare, which only recently has begun to concern many people (military planners, tacticians) throughout the military world. In this thesis we describe some of the anticipated operational contingencies of (tactical) missile warfare. We then formulate them as dynamic decision models solvable by techniques of operations research, and solve them. Before presenting the kind of problems which are to be analyzed in this thesis, we briefly explain what there is in air warfare trends today that causes conventional missile warfare to emerge as a major concern.

The principal weapon system in today's Air-to-Ground (A/G) warfare is still, undoubtedly, the aircraft. However, some significant technological achievements of the last decade (mainly in the field of Radar Technology and Electronic Warfare) raised a new generation of Surface-to-Air Missile (SAM) systems. These systems are becoming serious obstacles to a full and effective utilization of the aircraft in the A/G warfare. The existence of such Ground-to-Air (G/A) defense systems can sometimes prohibit the use of the aircraft, at least at the initial phase of an armed conflict. The reason

for that is that the attrition rates expected as a result of the presence of these systems stands at such a high level that it would be very hard, due to psychological factors, to launch manned vehicles into such effectively defended areas.

Among the main properties which make some of the modern SAM systems very hard to overcome are:

(1) They are operationally autonomous. The operability of a battery does not depend on any other units of the defense system, since each unit, or battery, includes all components necessary for detection and acquisition of targets, and for controlling and firing missiles.

(2) The batteries are small, compact, hardly detectable.

(3) The batteries are very highly mobile.

(4) Very sophisticated detection sensors and avanced acquisition, control and ECM devices make the SAM batteries highly effective in a very high kill envelope.*

These developments in defense technology are very likely to intensify trends, which can already be noticed to exist today in the U.S. and in Europe, to develop long range, tactical, conventional missiles for massive uses, at least at the initial phase of conventional armed conflicts.

A brief description of the characteristics of tactical missiles follows. We emphasize that we refer here to missiles used for attacking ground targets, which are considered mainly as alternatives for the aircrfat, in a specific, tactical mission.

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*Examples for systems which can be characterized by (1)-(4) above are: Soviet SA-8, SA-6 systems; the German "Roland" SAM system; the French "Crotale" system.

(1) The missile is directed to point targets by highly sophisticated guidance and control systems (these systems may be passive, active, semi-active laser guided, or they may be of a Man-In-The-Loop type).

(2) The accuracy of the missiles is very high: Circular Probable Errors (CEP) of a few feet are quite typical, i.e., the missiles have almost pinpoint accuracy.

(3) The warhead is conventional.

(4) The maximum ranges of the missiles are long enough to make it possible to position the launchers (of the attacking missiles) at a considerable distance from any intercepting system for which the defender might call. In other words, the launchers (of the attacker) are assumed to be safe and practically invulnerable.

(5) The operations with the tactical missile are heavily supported by an advanced tactical intelligence system. Such networks, which are very likely to play a dominant role in future air warfare will enable the attacker to receive and use information on (almost) a real time basis.

Our main concern in this thesis is the following scenario: Suppose a group of targets of high military value is defended by a local system, consisting of a group of independent SAM batteries. We assume that the defense system is sufficiently solid so as to force the attacker to avoid deploying aircrafts as long as the defense remains highly effective. To carry out missions which are urgent, as the attack of the group of targets mentioned above may be, the attacker may decide to use long range tactical missiles. These missiles may be aimed at the defending targets (i.e., SAM batteries) as well as at the defended ones. They can be intercepted by defensive missiles, which can be launched by the SAM batteries. Thus we consider here a pure tactical missile warfare. The main questions

which this thesis is dealing with are questions of optimal allocation of the tactical missiles between the value targets and the defending targets. (We discuss criteria of optimality later.)

Our terminology thus distinguishes between two categories of targets: We use the term "primary" to describe targets, the destruction of which is considered as the ultimate goal of the attacker's mission. We define as "secondary" targets those targets which defend the primary ones. As connoted by their name, the "secondary" targets do not have inherent value, so that the question whether a given secondary target was destroyed or not is not weighed in the overall rating of mission success. This means that the objective functions taken for the allocation processes presented in this thesis, are always defined in terms of destroyed primary targets only, the number of destroyed secondary targets never being a factor. The only reason for the attacker to be sometimes willing to attack secondary targets is that by killing secondary targets the attacker improves the probability of missile survival, and so indirectly increases the effectiveness of missiles aimed at the primary targets.

It should be noted that the identification of a specific military target as 'primary' or 'secondary' is not a matter of a pure, factual judgment on the nature of a target. Rather, the categorization of a target should be derived from the specific tactical situation and from the goals defined by the decision maker in each specific case. It is clearly a

matter of subjective perspective. To give an example, consider a case in which the tactical missiles command is required to support ground forces by attacking some C^3 (control, command and communication) centers of the enemy. Suppose these centers are defended by SAM batteries. In that case, the C^3 centers are naturally the primary targets, and the SAM batteries are the secondary ones (since the judgment of mission success should be made from the perspective of the supported forces only, who quite expectedly will judge by evaluating damages to the C^3 centers only). On the other hand, there may be a case where the goal is to destroy a group of SAM batteries of one type, defended by SAM batteries of another type. Then the batteries of the first type will themselves be the primary targets.

The general optimal allocation problem with which we deal in Part I of this thesis is the following: The attacker has a given number of missiles. The defender defends a primary target with one or with several secondary targets. Probabilities of hitting the primary and secondary targets with one missile (assumed unintercepted by the defense) are given along with a probability of surviving an interception attempt made by the defense. The attacker launches his missiles sequentially, each missile aimed at either the primary target or a secondary target, according to the attacker's decision. The problem of the attacker is to find decision rules which maximize probability of killing the primary target (or, alternately, maximize the number of missiles penetrating into the

primary target). The important question is, under what conditions should he spend some of his missiles engaging secondary targets for the hoped-for benefit of improving the survivability (hence, the effectiveness) of the rest, which are to be aimed at the primary target? To deal with this kind of problem, we define two types of missiles:

(1) Anti-primary (AP) type: a type of missile which is designed to use against primary targets.

(2) Anti-secondary (AS) type: a type of missile which is designed to use against secondary targets.

We have made this distinction between AP and AS missiles because primary and secondary targets usually differ in their physical properties (hardness, size, shape, detectability by missile seeker, etc.), so that AP and AS missiles may differ in technical design (for example, they may have different warheads). It should be emphasized however that they are not necessarily different. It is possible that the two types will, in fact, be technically identical.

In Part II of the thesis we analyze a different concept of missile warfare--the concept of deceiving the secondary targets by decoys. We think of AS missiles as representing one concept of upgrading the survival rate of AP missiles (i.e., killing the defending targets) and of decoys as representing another concept of doing that (i.e., denying the secondary targets their ability to function effectively).

Decoys are "dummy" missiles which are designed to appear on the radar monitors of the secondary targets as real missiles. Decoys do not have warheads nor guidance heads. Some other

components of a real missile are either missing or technically simplified and less sophisticated in decoys. The decoy has an engine and some navigation devices to allow it to fly in a path similar to that of a real missile. Its physical signatures (RCS, Optical and IR signature) are very much the same as those of a real missile. Presumably, the decoy is much less expensive than a real missile, otherwise there is no reason to consider it.

The idea behind using decoys is that by prudently combining them with real missiles one can sometimes considerably save in mission cost while not losing much operational effectiveness (some effectiveness should however be lost, of course, whenever a real missile is replaced by a decoy). To put it differently, it is expected that decoys will improve the cost-effectiveness ratio. The goal of Part II of this thesis is to analyze situations in which such an improvement is indeed achieved.

B. METHODOLOGY

In this work we propose and solve models, to describe engagement processes, or duels, which take place between an attacker and a defender. The attacker owns a mixed stockpile of tactical missiles. The defender is defending one primary target with one or more secondary targets. Assumptions are set forth about the character of the duel, and especially about the defensive operational policy, the available information, the number of missiles the attacker may launch etc.

These assumptions lead to a formulation of mathematical models that are believed to reflect particular tactical situations which are of interest.

The above mathematical models are solved using recursive analytic methods such as stochastic dynamic programming, stochastic game theory and some notions from general decision theory.

Although the mathematical solving techniques are often essential to making our models usable, it is not the technical aspects which are of main concern, but rather the applicability of the models. Therefore, the general theories underlying the techniques are only briefly sketched wherever they are in use in this thesis. The literature relevant to each subject treated in this thesis is reviewed separately within the chapter in which the specific subject is presented.

It is important to emphasize here that the duels which are modeled in this work have to be regarded as processes belonging to the "microscopic" analysis of tactical missile warfare. Large operations with tactical missiles could adequately described as aggregates of many such duels. The method of this work is to isolate events or to focus on "atomic" conflicts. We pretend that in each such event an initial state is determined and both sides act to optimize some objective, which is a random quantity associated with the result of the duel.

C. STRUCTURE AND CONTENTS OF THE THESIS

The general structure of the thesis is depicted in the following scheme (see Fig. I.1). Part I of the thesis is dedicated to problems in which the choice presented to the attacker at each stage is between launching an anti-primary and launching an anti-secondary missile. In Chapters II and III, one-sided models are presented. In these models, it is assumed that each missile the attacker launches has a known and constant probability of survival, which depends on the number of secondary targets still alive. Probabilities of hit (conditioned on survival) are also given, and the problem is to find optimal decisions as a function of the number of secondary targets present and the number of missiles the attacker is allowed to launch. Three different objective functions are considered: (1) Probability of hitting the primary target; (2) Expected number of AP missiles hitting the primary target; and (3) Expected cost of hitting the primary target. (When using this last criterion, the attacker is assumed to have no limit on the number of missiles he has.) Some theorems concerned with the general structures of optimal policies are given, and detailed algorithms to find the parameters defining the optimal policies are given, along with some numerical examples.

Chater IV introduces a new element to the problems discussed in Chapters II and III. We add the assumption that the operator of the secondary target has the option of using a "cautious" mode of operation, which differs from the normal

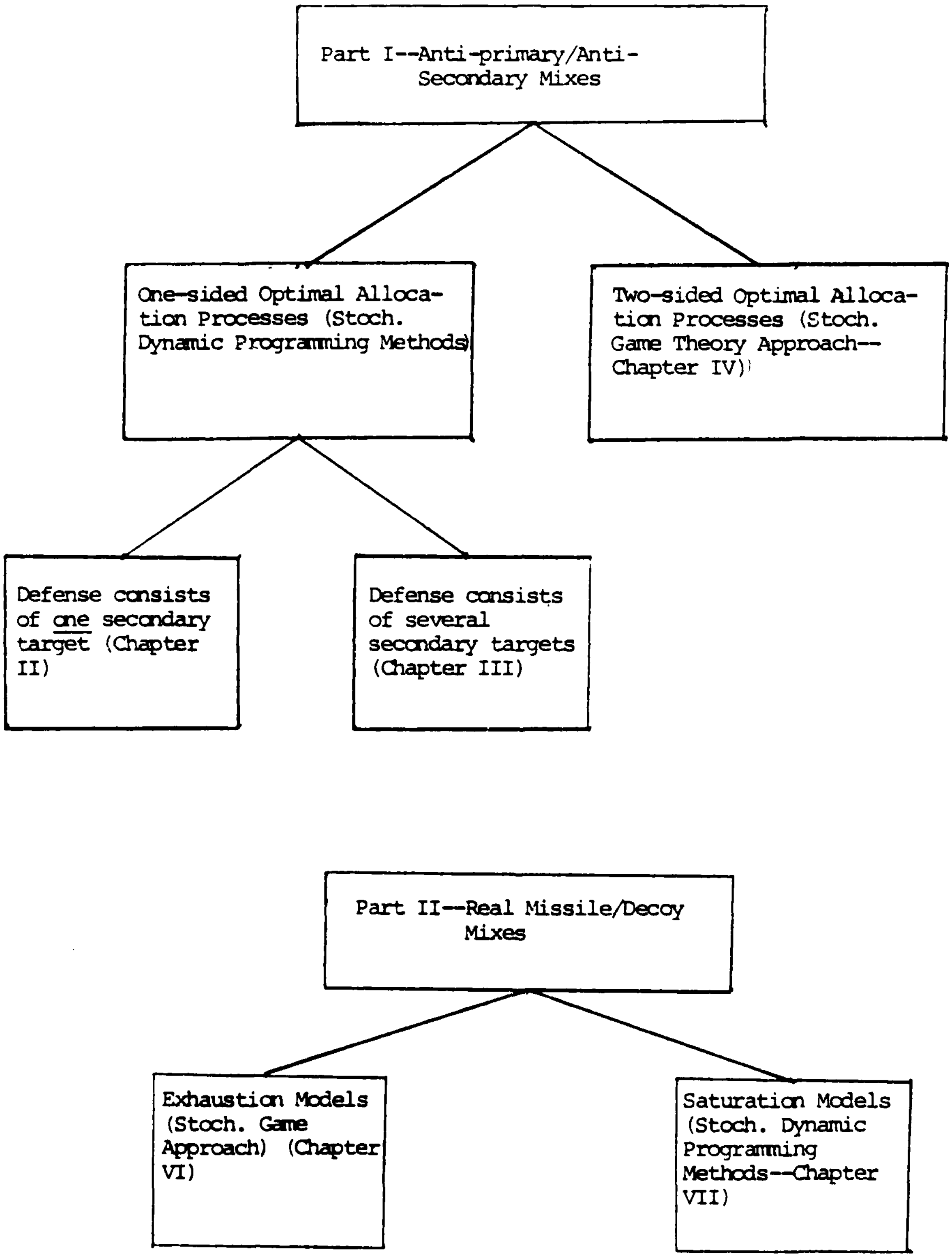

Figure I.1. Structure of the Thesis

mode in that it renders the secondary target invulnerable to AS-missile, and at the same time less effective against any threatening missile. The attacker and the defender, both aware of the options open to each other, are thus involved in a duel which is quite naturally modeled as a two person, zero-sum stochastic game. This game is fully analyzed and solved for the payoff functions mentioned above. A review of literature on stochastic game theory and its applications is given in Chapter IV.

Part II of the thesis is dedicated to problems in which a choice between launching a real missile and launching a decoy is made by the attacker in each stage of the process. Chapter V gives a short introduction to the concept of decoys, review of literature, and some presentation of the models which are treated in detail in the subsequent chapters. In Chapter VI we consider situations in which the mechanism by which decoys contribute to the operational effectiveness is exhaustion. A stochastic game is formulated where at each state the attacker may use a real missile or a decoy, and the defender may decide to fire one missile (from his limited stockpile) or to avoid firing. Costs are associated with the real missile and the decoy and the problems are to find min-max policies for various payoff functions.

In Chapter VII a different mechanism of decoy support is examined, which is the saturation mechanism. Here we assume that the attacker has the option of launching several missiles

(reals and decoys) simultaneously, where the defense can handle only one (or even more, but less than the number of missiles launched simultaneously) at a time. We find what should be the optimal number of decoys to accompany a real missile, and what is the optimal "mixture" of real missiles and decoys, so that the expected cost of killing the primary target is minimized. The technique used is that of stochastic dynamic programming.

D. SOME REMARKS ON THE USE AND APPLICABILITY OF THE MODELS PRESENTED IN THE THESIS

The models presented in this thesis are intended to aid a decision making process concerning the acquisition and deployment of conventional tactical missiles. In general, Operations Research models constitute only one block of a wider process of analysis, i.e., the "systems analysis" process. The term "systems analysis" usually describes a grand-optimization process, conducted at many levels of the defense organizational hierarchy of a nation, the goal of which is to suggest optimal courses of behavior. This process combines and weighs a broad scope of considerations, among them economic factors, political impacts, technological uncertainties, etc. One of the more important factors is of course the expected operational effectiveness of the system in various combat situations. The exploration of this must be done through <u>operational</u> <u>models</u>, models which attempt to foresee likely combat contingencies, and which are intended to become vehicles with which various quantitative questions (related to operational needs) are answered.

It is this last category of considerations within which this thesis is applicable. Our goal was to propose and solve models which seem adequate to various scenarios, and to suggest ways in which the results obtained through them can feed that broader process called "systems analysis." Our thesis is a pure "Operations Research" kind of work, and since much more should be considered at a "system" level, the reader cannot make quick conclusions regarding the tactical missile "system".*

Some remarks are necessary here on the relation between the models we propose in this thesis and the "real world" problems. Since we deal with future situations and make predictions--conjectures sometimes--of them, the term "real world" really deserves some clarification. We should bear in mind that we explore here processes that have not yet been experienced in the battlefield, and rely on some, perhaps optimistic, assumptions about future technological systems.

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*It is noteworthy that although a formal distinction between the disciplines of "Operations Research" and "System Analysis" hardly exists and is subject to many vague, ad-hoc interpretations, it is widely accepted that "Operations Research" represents the low level of analysis, whereas "systems analysis" represents the higher level. As Quade [6, p. 23] points out:

> "...when Operations Research came to be applied outside the military forces, the term was interpreted in its narrow sense, being confined to studies of low level problems where the decision maker had a clear objective in mind. The term systems analysis then began to be applied to broad "higher level" studies that looked into aspects that OR workers had

Hence, no certain facts about what can be viewed as "real world" and what can only be considered as an illusive or simplistic imagination, can be decided. The future reality may deviate from what we predict, and this deviation may be of such a degree so as to render some of our models inapplicable. This is, however, an inevitable deficiency of all operations research models related to future use of newly deployed weapon systems. This is precisely the same dilemma which exists in the analysis of anti-ballistic-missiles, or anti-submarine warfare, both of which have never been experienced before but have been extensively treated in the O.R. literature.

Using methodical terms, we can say that we lack here tools for validation of our models. With regard to the problem of validation, we quote from Fawcett [3, p. 13]. This reference is an excellent example of a work written with the same philosophy of research guiding this thesis. It deals with rational selection of tactics for A/G attacks in various situations. On the validation problem, Fawcett writes:

> ...A model would be accepted as valid if its structure parallels the real world situation of interest sufficiently accurately to allow useful conclusions to be drawn. The decision maker must decide whether or not this is true for a given model in a given decision situation. Accordingly, the validity of an operations research model is meaningful only when

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usually considered 'given' (the objective, for instance) and accepted models that seemed to some hardly scientific..."

"...To avoid confusion, it is suggested (as we do here) that the term "Operations Research" be confined to efficiency problems, and "system analysis" to problems of optimal choice."

a specific decision is to be made; otherwise, there is no basis for judging whether or not the model constitutes a satisfactory representation and the concept of validity has no meaning."

Our approach therefore was to propose and analyze models that, as anticipated today, grasp the essential relevant factors of tactical missile allocation processes.

Finally, it should be noted that although our interests lie in the area of tactical conventional missiles, much of the material here can be quite adequate to describe attack processes involving intercontinental ballistic missiles. In some models, only minor modifications should be made to make them adoptable for use in ballistic missile warfare research.

PART ONE: OPTIMAL ALLOCATION PROBLEMS INVOLVING ANTI-PRIMARY AND ANTI-SECONDARY MISSILES

II. THE OPTIMAL ALLOCATION PROBLEM WITH A SINGLE SECONDARY TARGET

In this chapter we present a basic allocation problem which, although simplified in many aspects, is still helpful in gaining insight into the military operational questions which are the theme of this thesis. Besides, the model presented in this chapter can very well be useful by itself as it fits directly some real tactical scenario.

In the model analyzed here we assume that one primary target is defended by a single secondary target. It is also assumed that the defender (who operates the secondary target) responds to all missile attacks with an attempt to intercept the offensive missile. In Chapters III and IV we elaborate on this model by considering the case in which several secondary targets defend the primary target (Chapter III) and by allowing a more sophisticated defense policy (Chapter IV).

A. DESCRIPTIONS OF THE BASIC MODEL

1. Assumptions

We assume that the attacker is allowed to make M attacks against a primary target, which is defended by a single secondary target. Due to some technical restrictions, or by command decision, the attacker is restricted to launch his missiles one by one. After every launch, the attacker is informed whether

the target at which he aimed his last missile was hit or missed. He uses this information (which he receives through the intelligence system, which is assumed ideal) to make the decision of which target (the primary or the secondary) should be the next one to be engaged.

The secondary target is always thought of as a Surface-to-Air Missile (SAM) battery of a technologically advanced type. Such a system has only one component the killing of which renders the whole system inoperable. This component is the control unit, which is usually a small vehicle or van, with the radar installed at the outside and the operators and control panels located inside. This control vehicle operates 3-6 launchers which are located aroung it.* There is no purpose in attempting to kill launchers because of their high redundancy.

The secondary target is therefore a 'point target' actually. In military terminology, a 'point target' is one with characteristic dimension which is much smaller than the radius of effectiveness of the weapon used against the target. (Thus being a 'point target' is not a physical feature of the target alone; it is determined by both the target and the weapon.)

We shall also assume that the primary target is a point target, although this assumption is not always necessary, as will be discussed later.

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*The Russian SA-6 system, or the French "Crotale" are examples for that type of defense system.

The practical significance of the assumption that a target is a point target is that the attack of the target by a missile can result either in a hit, and consequently in a complete destruction of the target, or in a miss, which means that no damage is inflicted upon the target. No result of intermediate significance is assumed to be possible (partial damage, for instance). Hence each attempt to destory the target with one missile is considered a Bernoulli trial, with a given probability of success (discussed below).

Another assumption which has to be explicitly stated, is that the primary and secondary targets are located sufficiently distant from each other so that it is impossible for a single missile to damage both of them. The distance is not so large, however, as to enable the defender, who uses the radar of the secondary target for detection of offensive missiles, to distinguish between missiles which are aimed at the primary target and missiles which are aimed at the secondary target.

The missiles to which we refer in this thesis are guided missiles of a very high accuracy of hit (Circular Probable Error [CEP] of 3-6 feet are considered typical). There is no need in this thesis to refer to specific features of the guidance system or of any other component of the missiles. All the parameters needed for the operations research analysis are assumed known to the attacker, and will be discussed later. In real applications of the methods offered in this thesis one should, of course, evaluate those parameters in order to apply

the model, and this evaluation in itself usually amounts to a very hard preliminary analysis.

It should be noted that the missile which the attacker uses against the primary target may or may not be of the same type of missile used to attack the secondary one. Different missiles would normally be required due to differences in the vulnerability and physical characteristics between the primary and the secondary target. But this is not essential to the models presented in this chapter. The missiles are launched from an airborne launcher or from a ground launcher. In all cases we assume that the missiles have ranges long enough so as to enable the attacker to locate the launchers in a safe place. In other words, all the models in this thesis assume that the launcher is practically invulnerable.

2. The Parameters of the Problem

We denote by q the probability that a missile launched at either the secondary or the primary target will survive an interception attempt made by the secondary target. In other words, the secondary target, so long as it is alive, has probability 1-q of intercepting an offensive missile while it is still flying to its target, thus preventing it from reaching the vicinity of the target.

We denote by P_p the probability that a single offensive missile, aimed at the primary target (AP-missile), will hit the target, given that it survives the secondary target. Thus, the overall probability of killing the primary target by an

AP-missile is P_p if the secondary target is already dead, and $P_p \cdot q$ if it is still alive.

We define a parameter P_s in a similar way: it is the probability of hitting the secondary target by an anti-secondary missile (AS-missile) once it survives the secondary missile.

The parameters P_p and P_s are sometimes called--in military terminology--the Single-Shot-Probabilities-Of-Kill (SSPK). They are of course functions of many factors such as the physical characteristics of the targets, technical features of the missile, launch tactics, environmental factors (weather, visibility), etc. As was explained before, the parameters should be estimated prior to implementation of our model by properly evaluating the effects of each of the above factors. It is clear, however, that in reality there is no hope of getting an "accurate" value of P_p, P_s and q. Therefore, a sensitivity analysis of the results should be carried out when implementing the models to be presented here.

3. Policies and Criteria of Effectiveness

As long as the secondary target is alive, the decision problem faced by the attacker at every stage is at which target to launch the next missile. Each stage is defined by the number of missiles left to be used (denote it by M). We denote by AP (AS) the decision to launch a missile at the primary (secondary) target. Let $\mathbb{N}$ be the set of positive integers. A 'policy' is a function D(M), D: $\mathbb{N} \to \{AP, AS\}$ which prescribes

which target should be selected for attack at any number of missiles left to be launched, assuming, of course, that the secondary target has not yet been destroyed. When the secondary target is dead, the decision problem is, of course, trivial. We denote by $\mathcal{D}$ the set of all policies.

In order for optimality concepts to be meaningful, we have to define a criterion of effectiveness of a policy. Two criteria of effectiveness are analyzed in this chapter.

a) Maximum-Probability-Of-Hit (MPH) Criterion: a policy determines the probability that the primary target will eventually be hit (given the number M of killing attempts left). We denote by P(M;D) the probability of hitting the primary target by M attempts, given that the policy D is used. The problem to solve is to find the policy D* such that

$$P(M;D^*) = \max_{D \in \mathcal{D}} P(M;D) = P(M)$$

b) Maximum-Expected-No.-Of-Penetrators (MENP) Criterion: In this case the objective is to bring into the primary targets as many missiles as possible. A penetrator is an AP-missile which survives the secondary target. If we denote by E(M;D), the expected number of penetrators when the attacker is using policy D, with M missiles left to be launched, the problem is to find the maximizing policy D* such that

$$E(M;D^*) = \max_{D \in \mathcal{D}} E(M;D) = E(M).$$

Criterion a) above is usually the appropriate one when the primary target is a 'point target', and when the only measure of success is the target being killed or not. Thus, the natural objective function in that case is the probability that at least one missile hits the primary target. Criterion b) is the more natural one when the primary target is not a single

'point target', but rather a compound of many small targets. Examples are: Airfields, industrial facilities, etc. In such cases, the relevant measure is the number of missiles, aimed at the primary target, which penetrates through the defense. The argument behind the adoption of such a criterion in this case is that the expected damage inflicted upon the target is perceived to be directly related to the number of penetrating AP-missiles.

A third criterion will also be examined in this thesis, but its treatment is postponed to the next chapter where the problem with that criterion is presented and solved in a more general situation, i.e., where the numbers of both primary and secondary targets are assumed arbitrary. To use this criterion, the model is slightly changed: Rather than assuming that the attacker is restricted to a given number of launches, we assume that he goes on with the attacks until the primary target is killed. The objective is to find a policy which minimizes the expected cost of achieving that kill (there are costs C_p and C_R associated with an AP- and an AS-missile, respectively). This criterion is abbreviated by MEC (Minimum-Expected-Cost).

In Section B we solve the problem with MPH criterion. In Section C the MENP criterion is treated. The problem with the MEC criterion is solved in a more general context, as explained above, in Section III.E.

B. THE ALLOCATION PROBLEM WITH THE MAXIMAL-PROB-OF-HIT (MPH) CRITERION

1. The Functional Equation

Let us denote by $Q(M)$ the probability of missing the primary target, when the attacker starts with M missiles and makes optimal decisions at each stage. It is also assumed that the secondary target is still alive. Clearly

$$Q(M) = 1 - P(M) = 1 - P(M;D^*)$$

where D^* is the optimal policy. If the secondary is dead, the probability of missing the primary target is denoted by $Q_0(M)$, and is given simply by

$$Q_0(M) = (1 - P_p)^M$$

We now write the stochastic functional equation for the function $Q(M)$. We use the dynamic programming principle of optimality due to R. Bellman [2]. The equation is:

$$Q(M+1) = \text{Min}\{P_s \cdot q(1-P_p)^M + (1-P_s q) \cdot Q(M), (1-P_p \cdot q) \cdot Q(M)\} \qquad \text{(II.1)}$$

The first term inside the brackets is the probability of not destroying the primary target, starting from state M+1, if the offender attacks the secondary target first and then proceeds optimally. The second term corresponds to a decision to attack the primary target first, and then proceed optimally.

To initiate the solution of the functional equation (II.1), we start from $M = 1$ and proceed forwards. To find $Q(1)$, notice

that when the offender is left with only one missile, the optimal decision is clearly to attack the primary target. Therefore we have:

$$Q(1) = 1 - P_p \cdot q$$

as before. Let $D^*(M)$ $(M = 1,2,\ldots)$ be the optimal policy. Then, $D^*(1) = AP$. Now let M^* be the smallest number such that $D^*(M^*) = AP$ whereas $D^*(M^*+1) = AS$. Thus, M^* is a number with the following property:

$$D^*(M) = AP \quad \text{for} \quad M = 1,2,\ldots,M^*$$

but

$$D^*(M^*+1) = AS.$$

The number M^* has the meaning that the attacker has to have at least M^*+1 missiles in order to afford spending at least one missile in the indirect action of attacking the secondary target. This does not exclude a priori the possibility that the optimal decision at some state $M > M^*+1$ would be to act against the primary target. In other words, it seems intuitively possible that for stockpiles of very large size M, and for at least some combination of the parameters P_p, P_s and q, it would be optimal to allocate the first missile to the primary target.

This intuition is, however, false. We show that if M^* is finite, then for all values of M greater than M^*, the

optimal decision is AS. However, it is possible that $M^* = \infty$, in which case the attacker should spend all his missiles on attacking the primary target, no matter how many he has (we show later that this happens when $P_p \geq P_s$).

This observation can be formulated as follows: In an optimal policy D^*, no switch is possible from attacking the primary target to attacking the secondary one. One switch at most is possible from the secondary to the primary target. We put this observation formally as a lemma.

2. <u>The Basic Lemma</u>

If D^* is optimal, and $D^*(M) = AP$ for some M, then $D^*(M') = AP$ for all $M' < M$.

<u>Proof</u>: We use a method of proof very common to problems of optimal sequential processes.* Let us assume, by contradiction, that D^* is an optimal policy, and that for some state $\tilde{M}$ we have:

$$D^*(\tilde{M}) = AP$$
$$D^*(\tilde{M}-1) = AS$$

Let us consider a different policy $D^{**}(M)$, defined as follows:

$$D^{**}(\tilde{M}) = AS$$
$$D^{**}(\tilde{M}-1) = AP$$

and

$$D^{**}(M) = D^*(M) \quad \text{for all} \quad M < \tilde{M}-1.$$

---

*See, for instance, R. Bellman [2, Ch. 2] for proof of optimal policy for the "Gold Mining" problem. This method of proof also applies to some problems in search theory and optimal scheduling problems.

D^{**} is almost identical with D^*. The two policies disagree only on states $\tilde{M}$ and $\tilde{M}-1$. Let us calculate now the values of the objective functions at state $\tilde{M}$ under policies D^* and D^{**}. We calculate this by conditioning on results of the first two stages (i.e., the first two missiles to be launched). We have:

$$Q(\tilde{M};D^*) \triangleq Q(\tilde{M}) = (1-P_p\cdot q)\cdot P_s\cdot q(1-P_p)^{\tilde{M}-2}$$
$$+ (1-P_p\cdot q)(1-P_s\cdot q)\cdot Q(\tilde{M}-2) \qquad \text{(II.2)}$$

$$Q(\tilde{M};D^{**}) = P_s\cdot q(1-P_p)^{\tilde{M}-1} + (1-P_s\cdot q)\cdot(1-P_p\cdot q)\cdot Q(\tilde{M}-2) \qquad \text{(II.3)}$$

Subtracting Eq. (II.3) from Eq. (II.2) we get:

$$Q(\tilde{M};D^*) - Q(\tilde{M};D^{**}) = P_sP_p\cdot q(1-P_p)^{\tilde{M}-2}(1-q) > 0.$$

Hence,

$$Q(\tilde{M};D^{**}) < Q(\tilde{M};D^*) = Q(\tilde{M})$$

Thus, D^* cannot be an optimal policy as was assumed. This completes the proof of the lemma!

3. Solution

The basic lemma shows that calculating M^* as a function of P_p, P_s and q is sufficient for completely specifying the optimal policy D^*. From the definition of M^*, we see (using Eq. (II.1) that:

$$Q(M) = (1-P_p\cdot q)Q(M-1) \quad \text{for} \quad M \leq M^*$$

so that

$$Q(M) = (1-P_p\cdot q)^M \quad \text{for} \quad M \leq M^*. \qquad \text{(II.4)}$$

M^* is known to be the smallest M such that the minimum in Eq. (II.1) is obtained by the first term, that is, M^* is the smallest M to satisfy the inequality:

$$P_s\cdot q(1-P_p)^M + (1-P_s\cdot q)Q(M) < (1-P_p\cdot q)Q(M)$$

or

$$P_s\cdot(1-P_p)^M < (P_s-P_p)Q(M) \qquad \text{(II.5)}$$

Substituting for Q(M), using Eq. (II.4), we get:

$$P_s\cdot(1-P_p)^M < (P_s-P_p)\cdot(1-P_p\cdot q)^M$$

or

$$\left(\frac{1-P_p}{1-P_p\cdot q}\right)^M < 1 - \frac{P_p}{P_s} \qquad \text{(II.5a)}$$

Now, since $q < 1$, we have $\frac{1-P_p}{1-P_p\cdot q} < 1$, so that the above inequality is equivalent to

$$M > \frac{\ln(1-\frac{P_p}{P_s})}{\ln(\frac{1-P_p}{1-P_p\cdot q})} \quad . \qquad \text{(II.5b)}$$

As implied by its definition, M* is the least integer which satisfies inequality (II.5b). Therefore we can now write:

$$M^* = 1 + \left[\frac{\ln\left(1-\frac{P_p}{P_s}\right)}{\ln\left(\frac{1-P_p}{1-P_p \cdot q}\right)} \right] \qquad \text{(II.6)}$$

where [x] denotes the greatest integer smaller than or equal to X. From inequality (II.5) we observe that if $P_s \leq P_p$, this inequality cannot be satisfied by any finite value of M*. Hence $M^* = \infty$, and this confirms a rather intuitively obvious fact that when the kill probability of the primary target (conditioned on survival of the missile) is equal or greater than that of the secondary target, it can never be desirable to allocate missiles to the secondary target.

Another interesting observation is the following: suppose we wish to know what should be the relation between the parameters P_p, P_s, q in order for M* to be equal to 1. We set:

$$\frac{\ln\left(1-\frac{P_p}{P_s}\right)}{\ln\left(\frac{1-P_p}{1-P_p \cdot q}\right)} < 1$$

and this implies:

$$1 - \frac{P_p}{P_s} \geq \frac{1 - P_p}{1 - P_p \cdot q}$$

or

$$1 + (P_s - P_p)q - P_s < 0.$$

Since we have assumed that $P_s > P_p$, we see that this last inequality cannot be satisfied by any triplet of values (P_p, P_s, q). In other words, the value of M* is at least 2 for all possible values of the parameters. Since when $P_s \leq P_p$ it is always optimal to launch at the primary target, we conclude that if the attacker has two missiles, he always has to allocate both of them to the primary target--no matter what are the values of P_p, P_s, q! This fact is somewhat surprising because intuitively one might very well suspect that for some combinations of values of P_p, P_s, q (for instance, for small q and P_p and large P_s) it would be better to use the first missile to attack the secondary target, in order to considerably improve the probability of hitting the primary with the second one.

Another question of a particular importance to operational analysis is the following: Given P_p, P_s and M, what values should q assume in order for it to be optimal to assign at least one missile to the secondary target?

This question can be answered by a straightforward argument based on inequality (II.5a) above. If--for particular values of M, q, P_p, P_s --the optimal decision is AS, then the four numbers M-1, q, P_p, P_s should satisfy inequality (II.5a), which can be shown to be equivalent to

$$q \quad q^* = \frac{1 - \dfrac{1-P_p}{\left(1-\frac{P_p}{P_s}\right)^{1/M-1}}}{P_p}$$

The optimal policy D^* (expressed as a function of q for given M, P_p, P_s) is thus:

$$D^* = \begin{cases} AP & \text{if} \quad q \geq q^* \\ AS & \text{if} \quad q < q^* \end{cases}$$

Notice that $q^* \to 1$ as $M \to \infty$ as expected. When the attacker has a large number of killing opportunities (and if $P_s > P_p$), we expect that it always is beneficial to start with the secondary target, unless q = 1, in which case all missiles surely survive so that there is no need to spend missiles on the secondary target. Notice also, that if M = 2, we get $q^* < 0$, and since q is a probability it can never be smaller than q^* in that case, so that the optimal decision for M = 2 is always AP--as discussed above. We now give an explicit expression for Q(M) for values of M greater than M^*. From the functional equation (II.1) we find:

$$Q(M) = P_s \cdot q(1-P_p)^{M-1} + (1-P_s \cdot q) \cdot Q(M-1) \qquad \text{(II.8)}$$

$$= P_s \cdot q(1-P_p)^{M-1} + (1-P_s \cdot q) \cdot [P_s \cdot q(1-P_p)^{M-2} + (1-P_s \cdot q)Q(M-2)]$$

$$= \ldots = P_s \cdot q \sum_{j=1}^{M-M^*} (1-P_s \cdot q)^{j-1} \cdot (1-P_p)^{M-j} + (1-P_s q)^{M-M^*} \cdot Q(M^*).$$

Noting that $Q(M^*) = (1-P_p \cdot q)^{M^*}$ (since for $M = M^*$ the optimal decision is AP, and it stays AP for $M < M^*$), we find:

$$Q(M) = P_s \cdot q(1-P_p)^{M-1} \cdot \sum_{j=1}^{M-M^*} \left(\frac{1-P_s \cdot q}{1-P_p}\right)^{j-1} + (1-P_s \cdot q)^{M-M^*} \cdot (1-P_p \cdot q)^{M^*} \qquad \text{(II.9)}$$

$$= P_s \cdot q(1-P_p)^{M-1} \left[\frac{1-\left(\frac{1-P_s \cdot q}{1-P_p}\right)^{M-M^*}}{1-\left(\frac{1-P_s \cdot q}{1-P_p}\right)}\right] + (1-P_s \cdot q)^{M-M^*}(1-P_p \cdot q)^{M^*}$$

$$= \left[\frac{P_s \cdot q(1-P_p)^{M^*}}{P_s \cdot q-P_p}\right] \cdot (1-P_p)^{M-M^*} + \left[(1-P_p \cdot q)^{M^*} - \frac{P_s \cdot q \cdot (1-P_p)^{M^*}}{P_s \cdot q-P_p}\right](1-P_s \cdot q)^{M-M^*}$$

In all the calculations which have been done here, we assumed $P_s \cdot q \neq P_p$. (The case $P_s \cdot q = P_p$ does not pose any special difficulty. It makes the summation in Eq. (II.9) simpler.)

From the last equation we see that Q(M) can be expressed as

$$Q(M) = A \cdot (1-P_p)^{M-M^*} + B \cdot (1-P_s \cdot q)^{M-M^*} \qquad \text{(II.10)}$$

where A and B are coefficients which depend on the parameters of the problem only (but not on M), and are given by:

$$A = \frac{P_s \cdot q(1-P_p)^{M^*}}{P_s \cdot q-P_p}, \quad B = (1-P_p \cdot q)^{M^*} - \frac{P_s \cdot q(1-P_p)^{M^*}}{P_s \cdot q-P_p}$$

Notice that we could arrive at the expression (II.8) (and hence, at Eq. (II.10)), by applying a direct probabilistic argument: We would condition the probability of miss on the number of missiles which are spent before a hit at the secondary target is achieved. The first term in the last row of Eq. (II.8) accounts for all the cases in which the secondary target is destroyed by one of the first M-M* missiles. The second term corresponds to the case in which all first M-M* missiles fail to hit the target, thus imposing on the attacker the necessity to switch and attack the primary target with all the remaining M* missiles.

C. THE ALLOCATION PROBLEM WITH MAXIMAL EXPECTED NO. OF PENETRATORS (MENP) CRITERION

1. The Functional Equation

Let E(M) be the optimal expected number of penetrators, given that the attacker is allowed to launch M missiles, and that the secondary target is alive. A penetrator is an AP missile which survives the defense (i.e., the secondary target) and thus penetrates into the primary target. If the secondary target is dead the attacker will of course launch all his remaining missiles at the primary target, and they are free to penetrate. The expected number of penetrators will be equal to the number of remaining missiles.

Suppose the secondary target is alive, and the attacker has M+1 missiles. If he launches the first one at the secondary target and then behaves optimally, the expected number of penetrators will be:

$$M \quad (\text{with prob. } P_s \cdot q)$$

or

$$E(M) \quad (\text{with prob. } 1-P_s \cdot q)$$

so that the unconditional expected number of penetrators will be

$$P_s \cdot q \cdot M + (1-P_s \cdot q)E(M) .$$

If, on the other hand, the attacker uses the first missile to attack the primary target, and then behaves optimally, the expected number of penetrators will be

$$q + E(M) .$$

The functional equation for E(M) is thus:

$$Q(M+1) = \text{Max}\{P_s \cdot q \cdot M + (1-P_s \cdot q)E(M), q+E(M)\} . \qquad (\text{II.11})$$

If the maximum in Eq. (II.11) is attained by the first term, the optimal decision is AS, and if it is attained by the second, the optimal decision is AP.

2. Solution

It can be shown, by much the same way it was done in Section II.B.2 that using the optimal policy, the attacker cannot switch from attacking the primary target to attacking the secondary one. Once the attacker directs a missile against the primary target, he should do so with all the rest.

Thus, the number M^* is defined here in exactly the same way as it was done before (Section II.B). For $M \leq M^*$ we get

$$E(M+1) = q + E(M)$$

from which we get the relation

$$E(M) = q \cdot M . \qquad \text{(II.12)}$$

To find M^* we use the fact that M^* should be the least value of M for which the first term in the functional equation (II.11) is greater than the second term. Thus, M^* is the least value of M to solve the equation:

$$P_s \cdot q \cdot M + (1-P_s \cdot q)E(M) > q + E(M) . \qquad \text{(II.13)}$$

For $M \leq M^*$, E(M) is given by Eq. (II.12), so that inequality (II.13) becomes, after substitution:

$$P_s \cdot q(1-q) \cdot M > q$$

Hence:

$$M^* = 1 + \left[\frac{1}{P_s(1-q)}\right] . \qquad \text{(II.14)}$$

It is now possible to express E(M), for $M > M^*$ by conditioning on the number of missiles that will be required to destroy the secondary target. With probability $(1-P_s \cdot q)^{j-1} \cdot P_s \cdot q$, the secondary target will be destroyed by the jth missile ($j = 1,2,\ldots,M-M^*$). In such a case, the remaining

M-j missiles will freely penetrate into the primary target. If all the first M-M* missiles miss the secondary target, the attacker will be left with M* missiles, and thus will have to switch to the primary target. The expected number of penetrators in that case is

$$E(M) = \sum_{j=1}^{M-M^*} (1-P_s\cdot q)^{j-1}\cdot P_s\cdot q(M-j) + (1-P_s\cdot q)^{M-M^*}\cdot M^*\cdot q$$

$$= M\cdot[1-(1-P_s\cdot q)^{M-M^*}] - P_s\cdot q\cdot \sum_{j=1}^{M-M^*} j\cdot(1-P_s\cdot q)^{j-1}$$

$$+ (1-P_s\cdot q)^{M-M^*}\cdot M^* q \ .$$

We now use the formula

$$\sum_{j=1}^{n} jX^{j-1} = \frac{1-(n+1)X^n + n\cdot X^{n+1}}{(1-X)^2}$$

to simplify the second term of the above equation. After some algebraic manipulation we finally get:

$$E(M) = M - \frac{1}{P_s\cdot q} + [\frac{1}{P_s\cdot q} - M^*(1-q)]\cdot(1-P_s\cdot q)^{M-M^*} \qquad \text{(II.15)}$$

We have chosen to express E(M) in the specific form given in Eq. (II.15) since it is the form which enables us, as we shall see later (Section III.D) to initiate an inductive proof of the general form for the expected number of penetrators when several secondary targets are present.

D. NUMERICAL RESULTS

Figures II.1 through II.7 refer to the allocation problem with MPH criterion (Section II.B). Figures II.1-II.3 can be considered as "decision charts", which can serve the attacker in immediately selecting, for each state and set of parameters, the optimal action. For any given values of P_p and M there corresponds a curve in the P_s-q plane, which separates the zone where the optimal decision is AP (the zone "above" the curve) from the zone where the optimal decision is AS (that which is below the curve). The curve itself simply gives the value of q* as a function of P_s.

Notice also, that the value of M*, for a given set of values P_p, P_s, q can very simply be discovered from the graphs given in Figs. II.1-II.3. To do that, one must select first the appropriate figure to use (according to the specific value of P_p). Then one should locate the point P_s-q on the figure and determine which is the closest curve, from below, to that point. The M-value of that curve is exactly M* for the combination of P_p, P_s, q examined. To clarify the last statement, let's find the M* value of the set

$$P_p = 0.2$$
$$P_s = 0.6$$
$$q = 0.5$$

From Fig. II.1, which is the one corresponding to the above value of P_p, we find that the point $(P_s,q) = (0.6,0.5)$ is

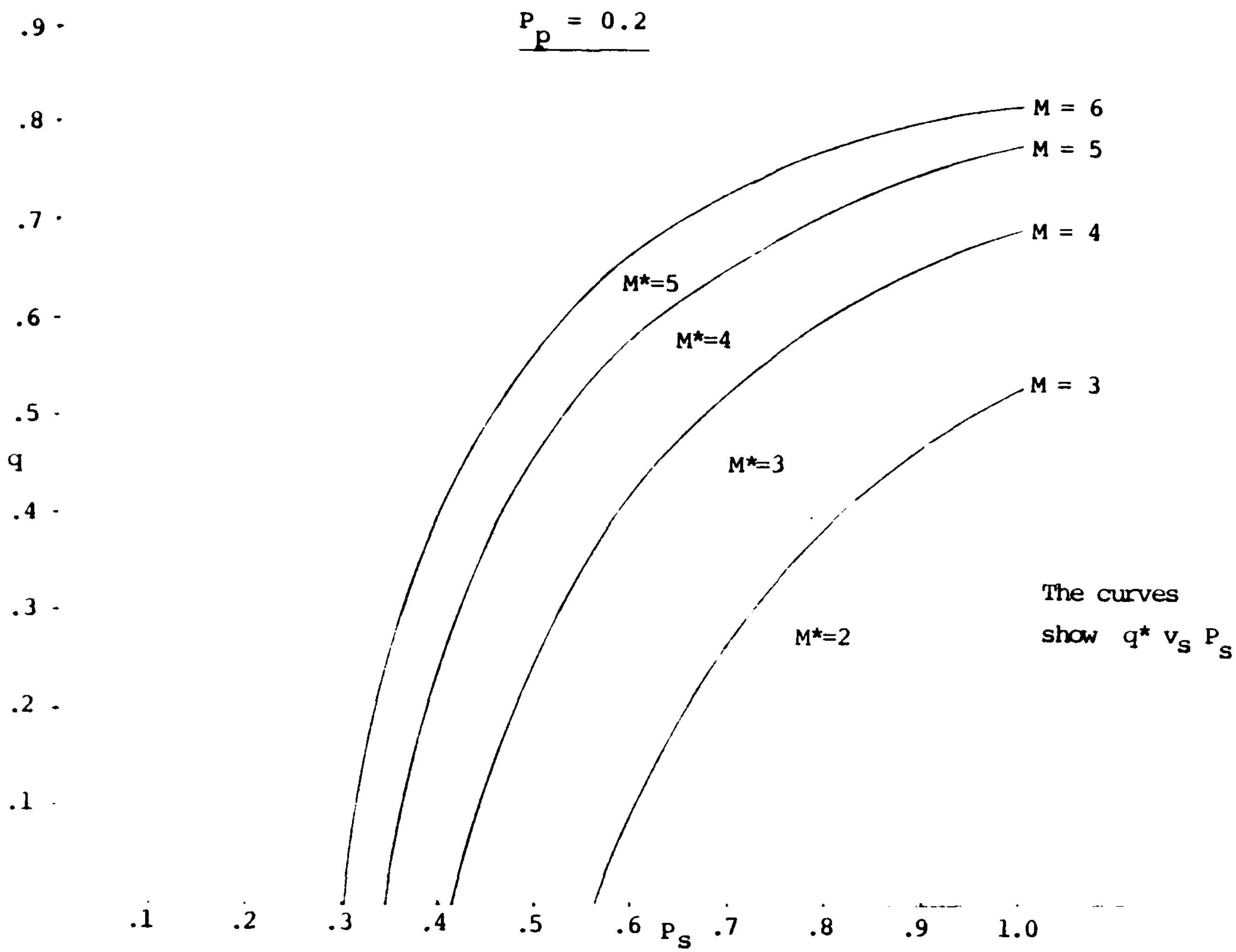

Figure II.1: Optimal Decision Curves for the Basic Allocation Problem (P_p = 0.2)

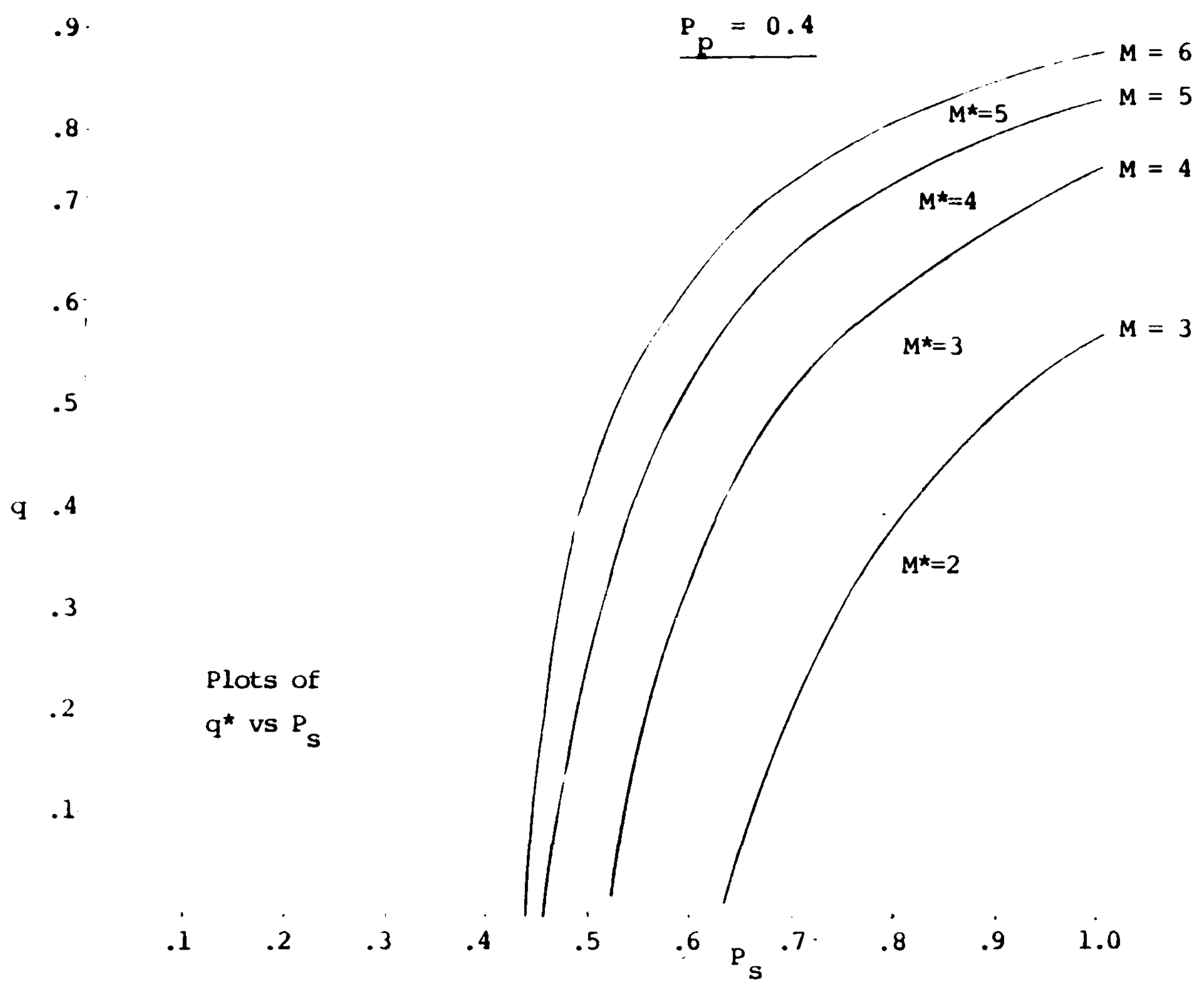

Fig. II.2: Optimal Decision Curves for the Basic Allocation Problem ($P_p = 0.4$)

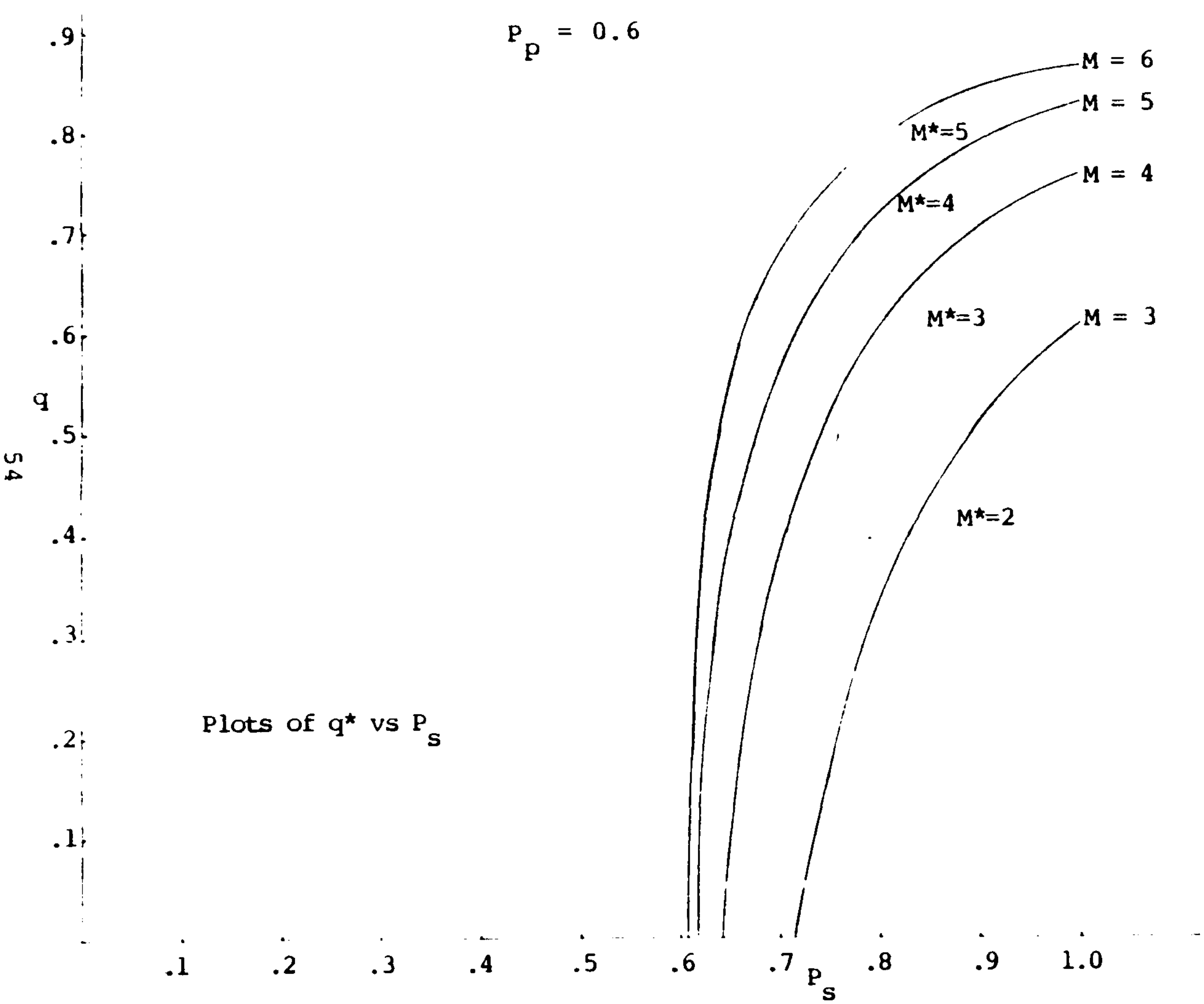

Figure II.3: Optimal Decision Curves for the Basic Allocation Problem (P_p = 0.6)

located between the curve with $M = 4$ and $M = 5$. This means that if $M = 4$, the attacker should take the AP decision, whereas if $M = 5$ he should take the AS decision. In other words, M^* is equal to 4.

In Figs. II.4 through II.7 we present the variation of the objective function $P(M)$ (i.e., the probability of hitting the primary target) with M, the number of missiles the attacker is allowed to launch. In order for the results to be indicative of the quantitative significance of using an optimal policy, we have chosen not to present the function $P(M)$ itself, but rather, the "scaled" probability of hit $J(M)$, which is defined by:

$$J(M) = \frac{P(M)}{1-(1-P_p \cdot q)^M} = \frac{1-Q(M)}{1-(1-P_p \cdot M)^M}$$

The function $J(M)$ is the ratio between the optimal probability of hit, and the probability of hit which will be achieved if the attacker uses the simple, "natural" policy of attacking only the primary target. As long as M is less than or equal to M^*, the optimal strategy itself is an "AP-only" strategy, and so $J(M) = 1$ for $M \leq M^*$, as seen in the graphs. For $M > M^*$, the function $J(M)$ is greater than 1, and has a maximum on some finite M. Also, notice that $J(M) \to 1$ as $M \to \infty$. The reason that $J(M)$ has that general form, which is seen in Figs. II.4-II.7, is very transparent: For very large values of M, the probability of killing the primary target becomes

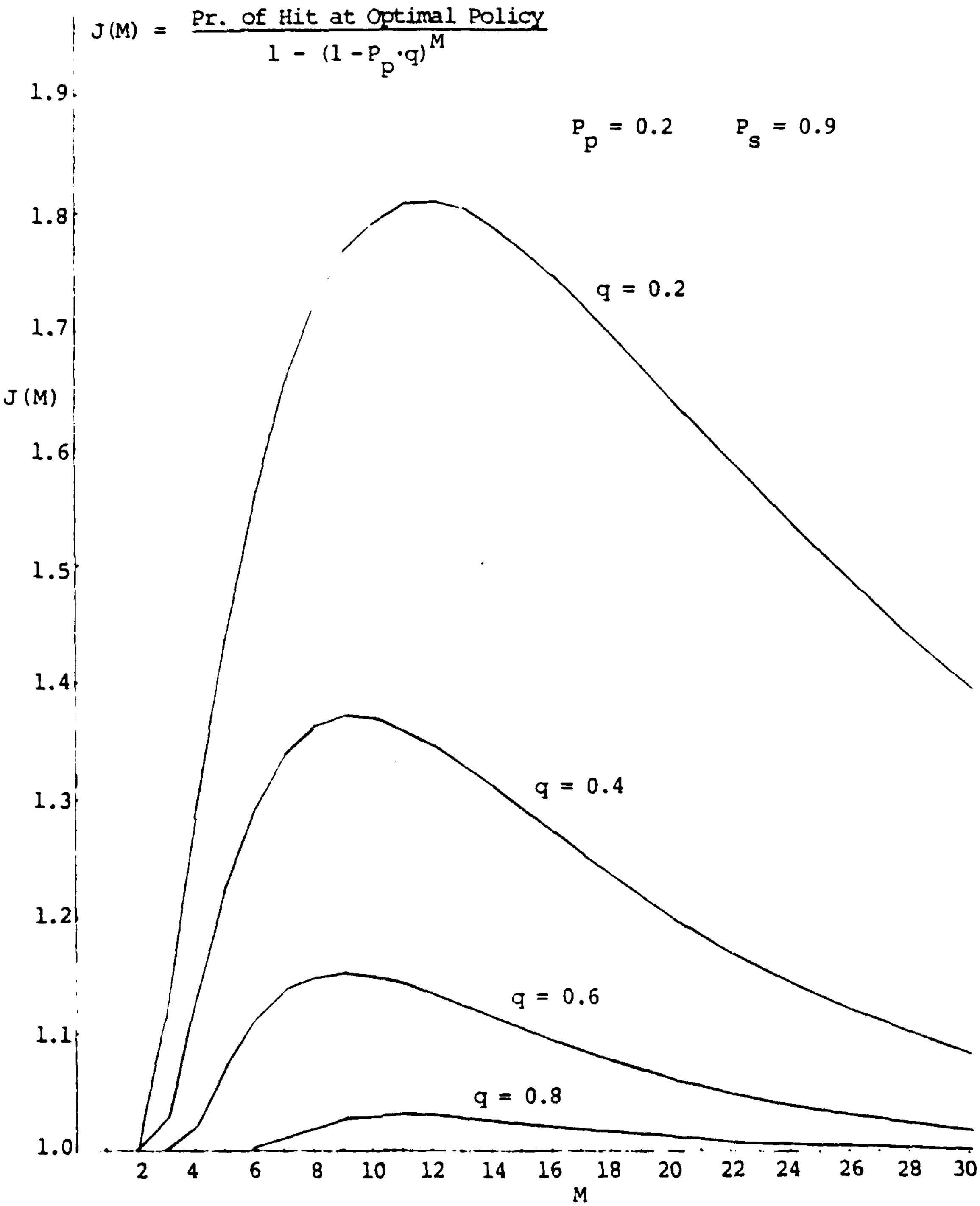

Figure II.4: Optimal (Modified) Objective Function J(M) for Allocation Problem with MPH Criterion, $P_p = 0.2$, $P_s = 0.9$

$$J(M) = \frac{\text{Pr. of Hit at Optimal Policy}}{1 - (1 - P_p \cdot q)^M}$$

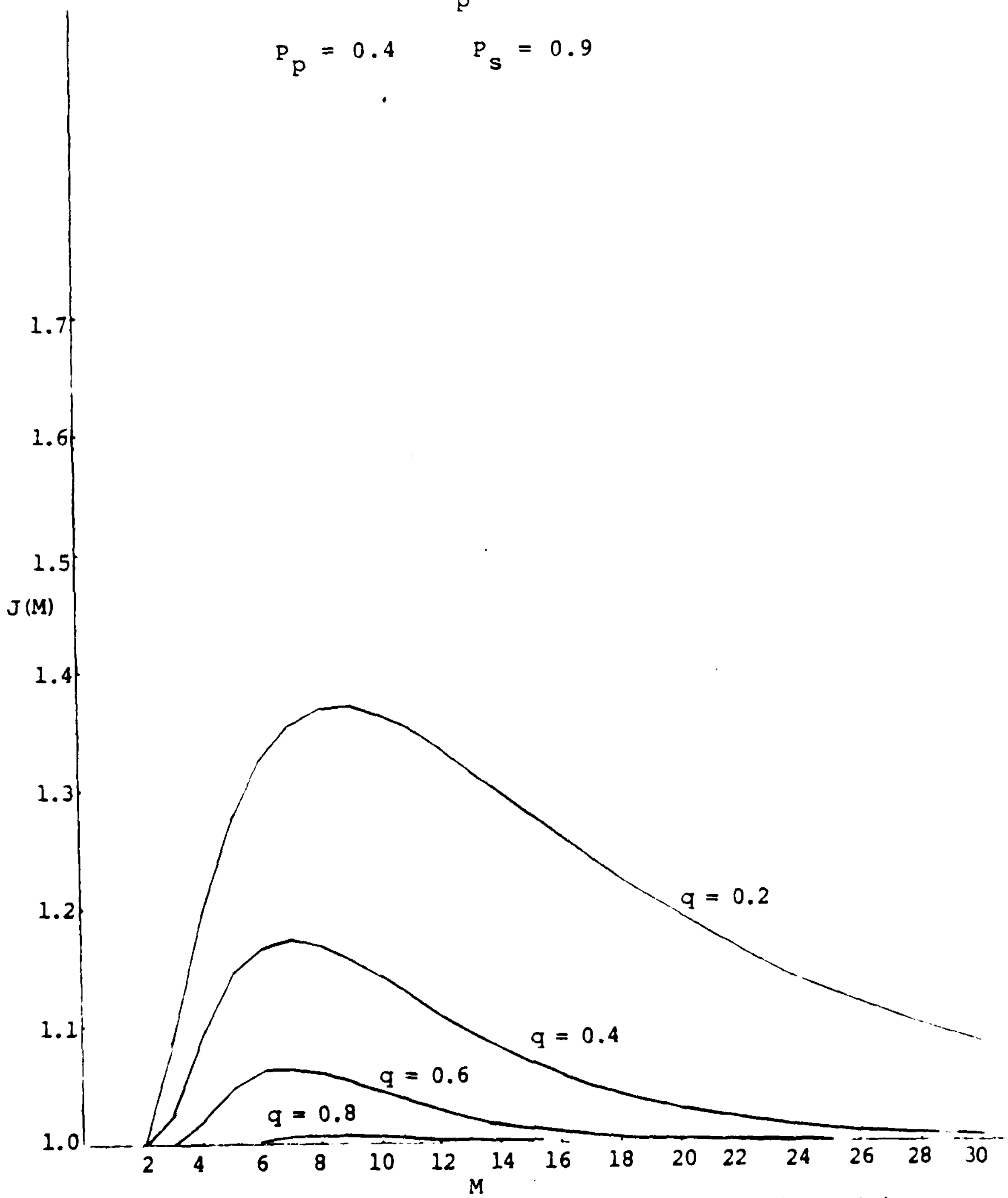

Figure II.5: Optimal (Modified) Objective Function J(M) for Allocation Problem with MPH Criterion, $P_p = 0.4$, $P_s = 0.9$

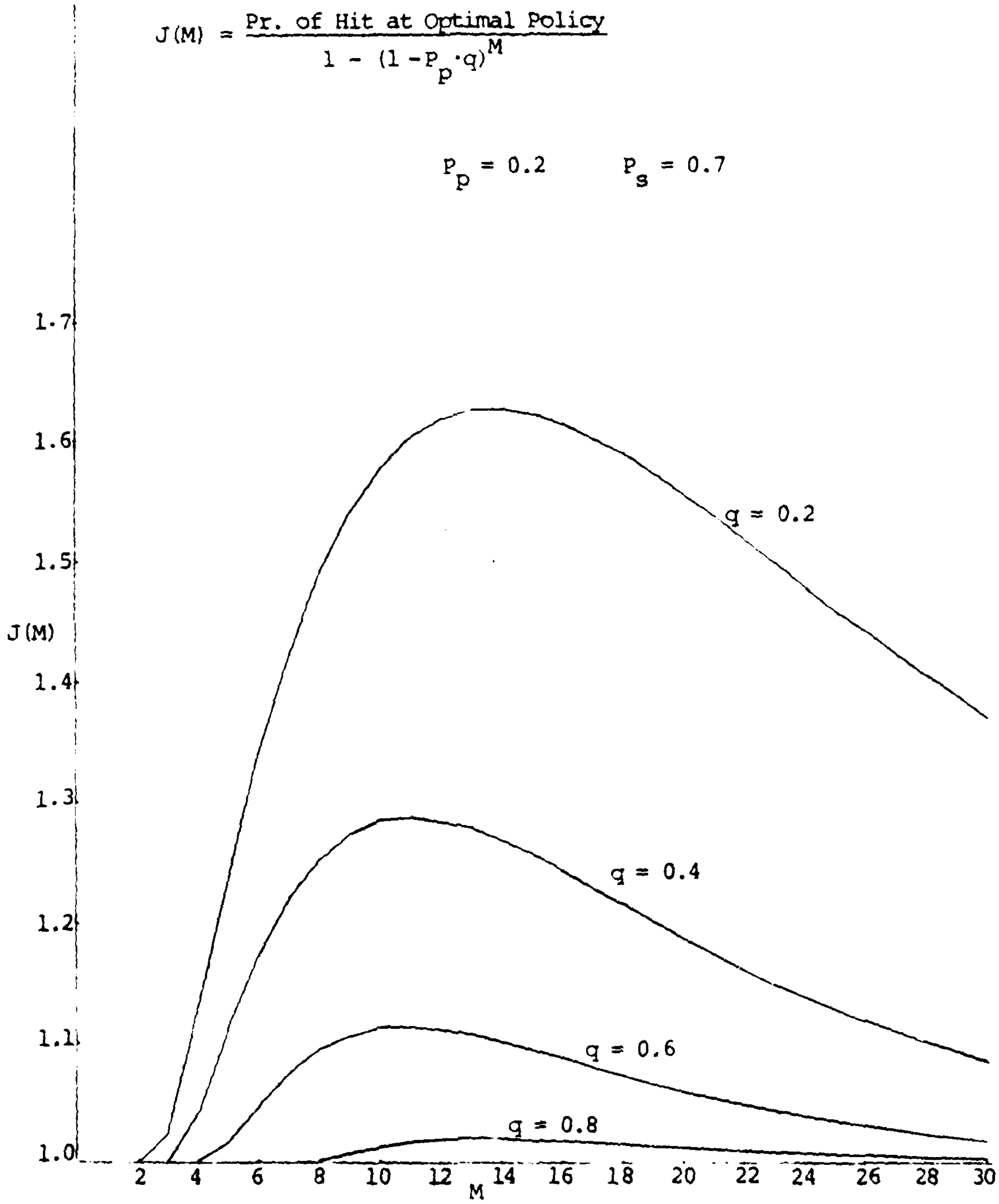

Figure II.6: Optimal (Modified) Objective Function J(M) for Allocation Problem with MPH Criterion, $P_p = 0.2$, $P_s = 0.7$

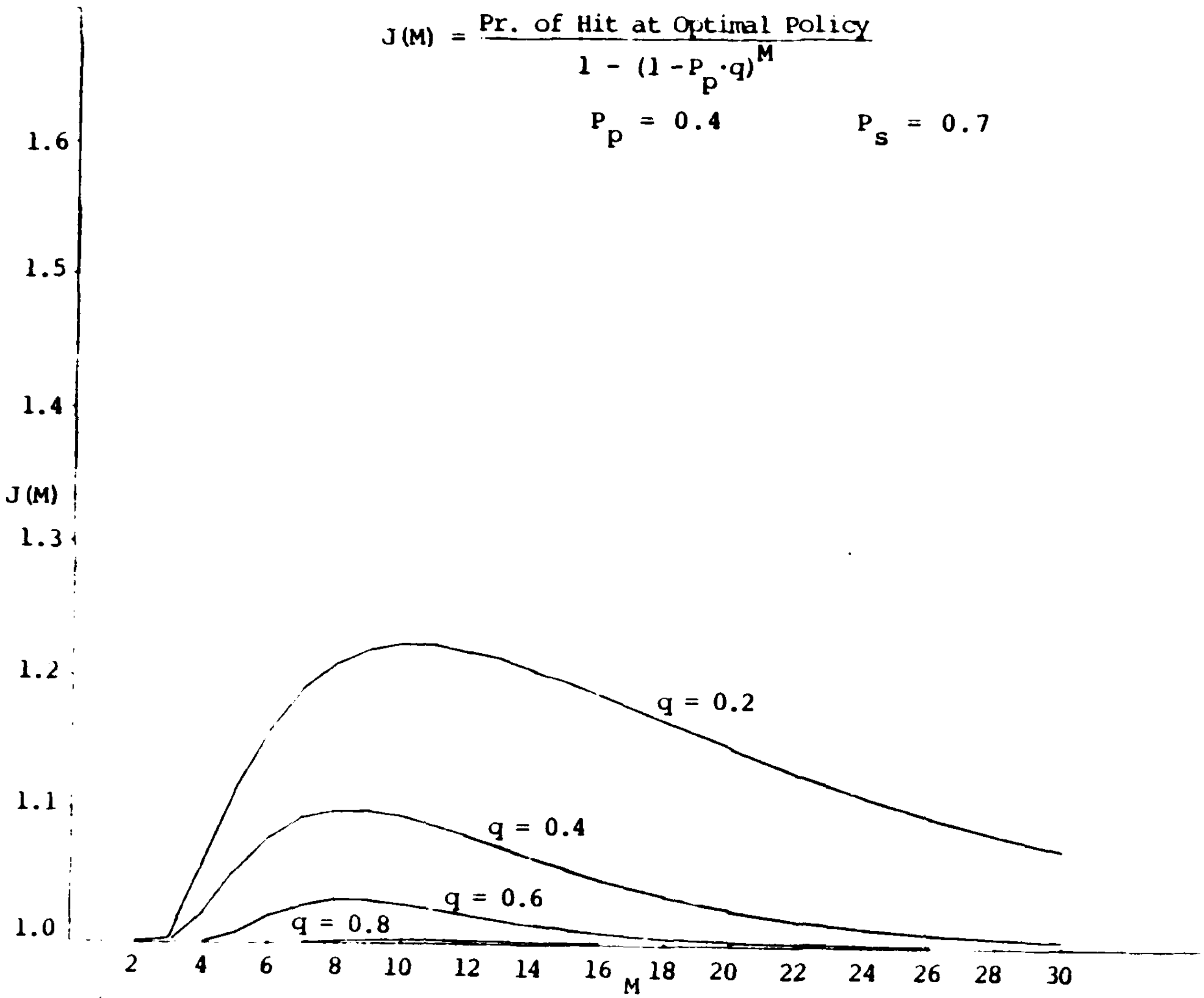

Figure II.7: Optimal (Modified) Objective Function $J(M)$ for Allocation Problem with MPH Criterion, $P_p = 0.4$, $P_s = 0.7$

very close to one even when using the non-optimal AP-only strategy. Therefore, $J(M)$ is very close to 1.

Notice that $J(M)$ directly gives the improvement achieved by the optimal policy, over what would be achieved in the non-optimal AP-only strategy. This improvement is seen to be more significant as the value of q gets greater. By comparing the various graphs we can also conclude that when P_p gets smaller (q and P_s fixed), or when P_s gets bigger (q and P_p fixed), the improvement $J(M)$ is more significant.

Figure II.8 refers to the allocation problem with MENP criterion. We present the function $H(M)$ defined by

$$H(M) = \frac{E(M)}{M \cdot q}$$

for $P_s = 0.7$ and for $q = 0.2, 0.4, 0.6$. Notice that the function $H(M)$ plays here the same role that $J(M)$ plays in the problem with the MPH criterion. The function $E(M)$ is given in Eq. (II.15) and is the optimal expected number of penetrators. The expression $M \cdot q$ in the denominator is the expected number of missiles which will penetrate if the attacker chooses to ignore the secondary target and to use the AP-only strategy. Thus $H(M)$ is the natural quantitative measure of the significance of using the optimal policy.

The function $H(M)$ is monotone increasing. It may easily be shown that $H(M)$ approaches q^{-1} as $M \to \infty$. This can be explained as follows: as M gets large, the expected number of penetrators gets very close to M. Using the "AP-only"

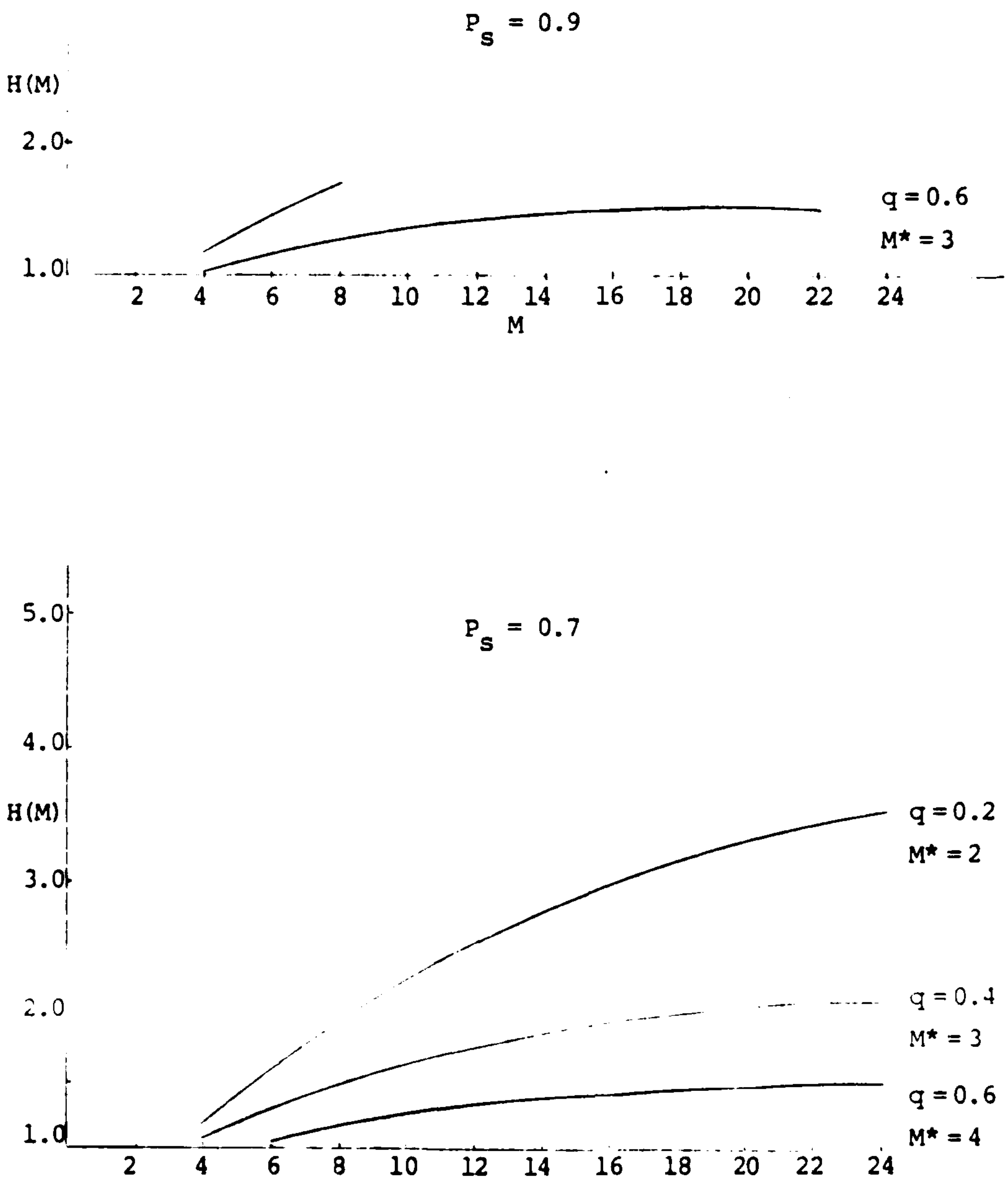

FIGURE II.8: Optimal (Modified) Objective Function H(M) for Allocation Problem with MENP Criterion

strategy the payoff is Mq. The ratio H(M) is thus very close to q^{-1}.

III. OPTIMAL ALLOCATION PROBLEMS WITH SEVERAL DEFENSIVE TARGETS

A. INTRODUCTION

In Chapter II we have worked on a problem in which only one secondary target was assumed to exist. In many instances, military targets of high value are defended by more than one secondary target. The effectiveness and survivability of the defense increases by adding secondary targets, whereas the survivability of offensive missiles decreases. In this chapter we analyze the same problem of Chapter II, but generalize it by allowing an arbitrary number of secondary targets to exist.

The assumptions we make are the following: We assume that all secondary targets are identical point targets. They are located in the vicinity of the primary target, close enough to the primary so that a detected offensive missile cannot be decided in advance on its destination (i.e., whether it is the primary or any of the secondary targets). The secondary targets are, however, sufficiently distant from one another so that a single missile cannot inflict damage to more than one target. We assume also that the destruction of one secondary target does not have any impact on the operability of other targets, that is, the individual targets are absolutely operationally autonomous.

Three different problems are solved in this chapter, corresponding to three different criteria of effectiveness:

(1) Maximizing probability of hitting the primary target (MPH criterion)

(2) Maximizing the expected number of anti-primary (AP) missiles which penetrate their target (MENP criterion)

(3) Minimizing expected cost of destruction of the primary target (MEC criterion).

The choice of the "appropriate" criterion to use depends on the specific military situation in which the models are to be implemented. We would use the MPH criterion if the primary target is a point target of high value, which practically can be assumed to exist in one of two states only: "killed" or "alive". The second criterion described above (MENP) fits a situation where the primary target is actually a big complex composed of many different point and area targets (e.g.: airfield, industrial facility). The level of damage to such a target can be one of a variety of partial, or intermediate levels (between the extremes of completely "killed" and completely "alive"). The number of AP missiles which penetrate such a target is usually quite adequate to represent the level of damage inflicted upon the target. The MEC criterion would be used in cases where the attacker is not limited in the number of missiles he can launch, so that the attack will be allowed to continue until the primary target is destroyed.

B. THE SURVIVAL FUNCTION

Let N_s be the number of secondary targets defending the primary target. We define a function, called the survival function, and denoted by $q(N_s)$, which is the probability that

an offensive missile will survive, given that N_s secondary targets are present. The function $q(N_s)$ is usually complicated and may be found from both experimental (evaluation tests) data and theoretical considerations. This function should satisfy the conditions:

(1) $0 \leq q(N_s) \leq 1$ (since $q(N_s)$ is a probability)

(2) $q(N_s) \leq q(N_s - 1)$ (since presumably, more defensive targets reduce the survivability of the attacking missile)

(3) $q(0) = 1$.

The form of the function $q(N_s)$ will usually be implied by the defensive strategy of controlling the operations of the N_s units. We give some examples:

(1) Independent Operations: Here we assume that upon any arrival of an offensive missile, each secondary target makes an attempt to intercept the missile, independently of all the others. In this case:

$$q(N_s) = q^{N_s} \quad \text{(where } q \text{ is a parameter)}$$

(2) Coordinated Operations: Here we assume that any potential direction of arrival of an offensive missile is protected by $n \geq 1$ SAM batteries if $n < N_s$ and by all N_s batteries if $N_s \leq n$. Thus,

$$q(N_s) = \begin{cases} q_0 = (\text{constant} = q^n) & \text{if} \quad N_s > n \\ q^{N_s} & \text{if} \quad N_s \leq n \end{cases}$$

(3) When operations are centrally controlled by a single unit, it is sometimes believed that the following process underlies the engagement phenomena. The decision taken by the defender is always to assign one unit to a detected offensive missile. However, there is a probability r that any single defense unit will be operative at the moment of its call, so that the defenders have a unit to assign only if not all the units are inoperative at the time when engagement is required. The probability of successful interception by any secondary target, given that it is operative, is assumed to be 1-q.

This leads to a survival function of the following form:

$$q(N_s) = (1-r)^{N_s} + [1-(1-r)^{N_s}]\cdot q$$

or

$$q(N_s) = q + (1-q)\cdot(1-r)^{N_s}.$$

The above are just the simplest types of survival functions which occur quite frequently. However, we shall not restrict ourselves to these special forms. Our goal is to investigate the structure of optimal assignment policy for a general survival function, with special emphasis on how this structure depends on general properties (such as convexity-concavity) of the function $q(N_s)$.

C. OPTIMAL ALLOCATION--MAX. PROB. OF HIT (MPH) CRITERION

1. Formulation

We say that the attack process is in state (N_s,M) if there are N_s secondary targets present and the attacker has

M missiles to launch. We denote by $Q(N_s,M)$ the probability of missing the primary target when starting from state (N_s,M) and using the optimal policy. The parameters P_s and P_p have the same meaning they had in Chapter II, namely, the probabilities of a surviving missile to kill the secondary and primary targets, respectively. The survival function is $q(N_s)$. The functional equation for the optimal probability of miss is:

$$Q(N_s,M) = \text{Min} \begin{cases} P_s \cdot q(N_s) \cdot Q(N_s-1,M-1) + (1-P_s \cdot q(N_s)) \cdot Q(N_s,M-1) \\ \\ (1-q(N_s) \cdot P_p) \cdot Q(N_s,M-1) \end{cases} \qquad \text{(III.1)}$$

The first (second) term is the probability of miss given that the first decision made by the attacker is to launch an AS (AP) missile, and then he uses the optimal policy. The rest of this section is dedicated to solve this equation and to analyze the structure of the optimal policy.

2. The Optimal Policy Structure--A Fundamental Lemma

A basic observation, which can be made without solving the functional equation, and which actually is the key to the solution process, is that if the attacker follows the optimal decision procedure, he can never launch an AS missile after at least one AP missile have been launched. That is, the optimal policy will always dictate to spend some missiles (and possibly none) on secondary targets and then to "switch" to the primary one, and use all the remaining launch opportunities to launch AP missiles. We present this statement as a

lemma and prove it by probabilistic methods. We denote by D^* the optimal allocation policy. Notice that any policy D is simply a function from the set of all states $S = \{(N_s,M): N_s, M = 1,2,\ldots\}$ to the set $\{AP,AS\}$, where AP (AS) means anti-primary (anti-secondary) decision.

<u>Lemma</u>: Let the survival function $q(N_s)$ be strictly decreasing. If for some state (N_s,M), the optimal policy D^* satisfies

$$D^*(N_s,M) = AP$$

then, for all values M', less than M, we have also:

$$D^*(N_s,M') = AP$$

<u>Proof</u>: We prove the lemma by contradiction. Suppose there is some state (call it $(\tilde{N}_s,\tilde{M})$), such that at the optimal policy we have:

$$D^*(\tilde{N}_s,\tilde{M}) = AP$$

and

$$D^*(\tilde{N}_s,\tilde{M}-1) = AS$$

Let D^{**} be another policy, which we now define by specifying the action it dictates on all possible states. We require:

$$D^{**}(\tilde{N}_s,\tilde{M}) = AS$$

$$D^{**}(\tilde{N}_s,\tilde{M}-1) = AP$$

and

$$D^{**}(\tilde{N}_s,\tilde{M}) = D^*(N_s,M)$$

for all $(N_s,M) \neq (\tilde{N}_s,\tilde{M})$ or $(\tilde{N}_s,\tilde{M}-1)$. Let $Q(N_s,M;D)$ denote the probability of miss achieved by using policy D, when starting from state (N_s,M). Then by definition of D* we have:

$$Q(N_s,M) = Q(N_s,M;D^*)$$

Repeating the same argument which was used in the proof of the lemma in Chapter II (page 40), we have

$$\begin{aligned} Q(\tilde{N}_s,\tilde{M}) &= Q(\tilde{N}_s,\tilde{M};D^*) \\ &= [1-q(\tilde{N}_s)\cdot P_p]\cdot q(\tilde{N}_s)\cdot P_s\cdot Q(\tilde{N}_s-1,\tilde{M}-2;D^*) \qquad \text{(III.2)} \\ &\quad + [1-q(\tilde{N}_s)\cdot P_p]\cdot[1-q(\tilde{N}_s)\cdot P_s]Q(\tilde{N}_s,\tilde{M}-2;D^*) \end{aligned}$$

and

$$\begin{aligned} Q(\tilde{N}_s,\tilde{M};D^{**}) &= q(\tilde{N}_s)\cdot P_s\cdot[1-q(\tilde{N}_s-1)\cdot P_p]\cdot Q(\tilde{N}_s-1,\tilde{M}-2;D^{**}) \\ &\quad + [1-q(\tilde{N}_s)\cdot P_s]\cdot[1-q(\tilde{N}_s)\cdot P_p]\cdot Q(\tilde{N}_s,\tilde{M}-2;D^{**}) \end{aligned} \qquad \text{(III.3)}$$

We now subtract Eq. (III.3) from Eq. (III.2), and use the following identities (which are implied by the fact that D* and D** are identical on $M < \tilde{M}-1$):

$$Q(\tilde{N}_s,\tilde{M}-2;D^*) = Q(\tilde{N}_s,\tilde{M}-2;D^{**})$$

and

$$Q(\tilde{N}_s-1,\tilde{M}-2;D^*) \;=\; Q(\tilde{N}_s-1,\tilde{M}-2;D^{**})$$

We then get:

$$Q(\tilde{N}_s,\tilde{M};D^*) - Q(\tilde{N}_s,\tilde{M};D^{**})$$

$$= \; q(\tilde{N}_s)\cdot P_s\cdot P_p\cdot[q(\tilde{N}_s-1)-q(\tilde{N}_s)]\cdot Q(\tilde{N}_s-1,\tilde{M}-2;D^*) \; > \; 0$$

where the last inequality comes from the assumption that $q(N_s)$ is strictly decreasing, and that $Q(N_s,M;D)$ is always positive. Thus,

$$Q(\tilde{N}_s,\tilde{M};D^{**}) \;<\; Q(\tilde{N}_s,\tilde{M};D^*)$$

and this contradicts the assumption that D* is the optimal policy! The lemma is thus provėn.

If $q(N_s)$ is not strictly decreasing, as was assumed, but simply non-increasing (as it must be!) the statement of the lemma should be slightly modified. From the proof of the lemma we see that if $q(\tilde{N}_s) = q(\tilde{N}_s-1)$, then policies D** and D* give identical probabilities of miss. Thus, we can only say that there always <u>exist</u> an optimal policy such that $D^*(N_s,M) = AP$ implies $D^*(N_s,M') = AP$ for all $M' < M$. By the discussion above we see that the only way to change optimal policies without destroying optimality is to reverse their dictations on states $(\tilde{M},\tilde{N}_s)$, where the value $\tilde{N}_s$ is such that $q(\tilde{N}_s) = q(\tilde{N}_s-1)$ (assuming this reverse changes the policy). We will be interested however only in policies which do not switch from AP decisions back to AS decisions. It is clear

that for such optimal policies, there is a value of M (denote it $M^*(N_s)$) associated with every value of N_s, such that the optimal policy (D^*) can be written as:

$$D^*(N_s, M) = \begin{cases} AP & \text{if} \quad M \leq M^*(N_s) \\ AS & \text{if} \quad M > M^*(N_s) \end{cases}$$

The function $M^*(N_s)$ thus completely characterizes the optimal policy D^*. It depends on the function $q(N_s)$ and on the parameters of the problem. In the next section we prove a theorem which is of great help in gaining some insight into the interrelation between the functions $q(N_s)$ and $M^*(N_s)$.

3. Non-Increasing M*-Sequences--The Monotone Miss Probability Ratio (MMPR) Concept

Let $\mathbb{N}$ be the set of natural numbers. We define the following function r(n) on $\mathbb{N}$:

$$f(n) = \frac{1 - P_p \cdot q(n)}{1 - P_p \cdot q(n-1)}$$

We call this function the "Miss Probability Ratio" (MPR). It gives the ratio of the probability of missing the primary target when n secondary targets are present to the probability of missing it when n-1 secondary targets are present. It can be viewed as a measure of the marginal reduction of the vulnerability of the primary target caused by the nth secondary target.

We say that the allocation problem has the Monotone Miss Probability Ratio (MMPR) property on the set $\{n: 1 \leq n \leq N\}$,

if the function $r(n)$ is monotone increasing on that set, that is, if

$$r(n) \geq r(n-1)$$

for every n in the set. We now state and prove a fundamental theorem which relates the function $M^*(n)$ to the survival function $q(n)$.

<u>Theorem 1</u>: A sufficient condition for the function $M^*(n)$ to be non-increasing on the set $\{n: 1 \leq n \leq N\}$ (for some N) is that the problem will have the MMPR property on that set.

Proof: We begin with some preliminaries. First we observe that the condition $M^*(n) < M^*(n+1)$ is equivalent to the condition

$$D^*(n+1, M^*(n)+1) = AP \quad . \qquad \text{(III.4)}$$

To show this, assume first that $M^*(n) < M^*(n+1)$. Then $M^*(n)+1 \leq M^*(n+1)$ so that by definition of $M^*(n+1)$, the optimal decision at state $(n+1, M^*(n)+1)$ must be AP. To show that the converse is also true, assume that Eq. (III.4) is valid. It implies the relation $M^*(n)+1 \leq M^*(n+1)$, so that $M^*(n) < M^*(n+1)$. The above mentioned equivalence is thus proven.

We now seek for an equivalent mathematical expression for the fact expressed in Eq. (III.4). To accomplish this we rewrite Eq. (III.1) with n+1 and $M^*(n)+1$ replacing n and M. Thus we have:

$$Q(n+1,M^*(n)+1) = \mathrm{Min} \begin{cases} P_s \cdot q(n+1) Q(n,M^*(n)) + (1-P_s \cdot q(n+1)) \\ \quad \cdot Q(n+1,M^*(n)) \\ \\ (1-q(n+1)P_p) Q(n+1,M^*(n)) \end{cases} . \qquad \text{(III.5)}$$

Since the second term in the brackets corresponds to an anti-primary first decision, it is clear that Eq. (III.4) is equivalent to that term being smaller than the first one. Hence,

$$(1-q(n+1) \cdot P_p) Q(n+1,M^*(n))$$

$$< \quad P_s \cdot q(n+1) Q(n,M^*(n)) + (1-P_s \cdot q(n+1)) Q(n+1,M^*(n))$$

This implies that the following inequality is equivalent to $M^*(n) < M^*(n+1)$:

$$Q(n+1,M^*(n)) \quad < \quad \frac{P_p}{P_s - P_p} \cdot Q(n,M^*(n)) \qquad \text{(III.6)}$$

In state $(n,M^*(n))$ the optimal decision is AP, and it stays AP through the whole process since no switch from AP to AS is possible. Hence

$$Q(n,M^*(n)) \quad = \quad [1 - P_p \cdot q(n)]^{M^*(n)} .$$

Furthermore, since $M^*(n) < M^*(n+1)$, we deduce that in state $(n+1,M^*(n))$ the optimal decision is also AP, and stays so. Thus:

$$Q(n+1,M^*(n)) \quad = \quad [1 - P_p \cdot q(n+1)]^{M^*(n)} .$$

Substituting these expressions of $Q(n,M^*(n))$ and $Q(n+1,M^*(n))$ in Eq. (III.6) we find:

$$\left(\frac{1-P_p\cdot q(n+1)}{1-P_p\cdot q(n)}\right)^{M^*(n)} < \frac{P_s}{P_s - P_p}$$

or:

$$M^*(n)\cdot \ln\left(\frac{1-P_p\cdot q(n+1)}{1-P_p\cdot q(n)}\right) < \ln\left(\frac{P_s}{P_p - P_p}\right) \quad , \qquad \text{(III.7)}$$

Thus we have shown that inequality (III.7) is equivalent to $M^*n) < M^*(n+1)$. We prefer to emphasize the equivalence of the converses, namely, that the relation $M^*(n) \geq M^*(n+1)$ is equivalent to:

$$M^*(n) \geq \frac{\ln\left(\frac{P_s}{P_s - P_p}\right)}{\ln\left(\frac{1-P_p\cdot q(n+1)}{1-P_p\cdot q(n)}\right)} \quad . \qquad \text{(III.8)}$$

Notice now that $M^*(1)$ was already calculated, and is given by Eq. (II.6). We have:

$$M^*(1) = 1 + \left[\frac{\ln\left(1-\frac{P_p}{P_s}\right)}{\ln\left(\frac{1-P_p}{1-P_p\cdot q(1)}\right)}\right] > \frac{\ln\left(1-\frac{P_p}{P_s}\right)}{\ln\left(\frac{1-P_p}{1-P_p\cdot q(1)}\right)} \qquad \text{(III.9)}$$

If we put now $n = 1$ in (III.8) and then replace $M^*(1)$ in the left-hand side with the smaller quantity given in the right-hand side of Eq. (III.9) we arrive at a condition which is sufficient (only!) for $M^*(1) \geq M^*(2)$. The condition is

$$\frac{\ln\left(1-\frac{P_p}{P_s}\right)}{\ln\left(\frac{1-P_p}{1-P_p\cdot q(1)}\right)} \geq \frac{\ln\left(\frac{P_s}{P_s-P_p}\right)}{\ln\left(\frac{1-P_p\cdot q(2)}{1-P_p\cdot q(1)}\right)} ,$$

or:

$$\frac{1-P_p\cdot q(1)}{1-P_p} \leq \frac{1-P_p\cdot q(2)}{1-P_p\cdot q(1)} ,$$

that is:

$$r(1) \leq r(2) .$$

Thus, we have shown that a sufficient condition for $M^*(1) \geq M^*(2)$ is $r(1) \leq r(2)$, which is exactly the assertion of the theorem for $N = 2$. We proceed by induction on N. To carry out the induction step, we need to use one more result which is an immediate by-product of the analysis made so far. We show that if it is known that $M^*(n+1) \leq M^*(n)$, then $M^*(n+1)$ can be exactly calculated. First notice that if we put an arbitrary M in place of $M^*(n)$ in Eq. (III.5), then by definition $M^*(n+1)$ would be the smallest integer for which the first term in brackets is smaller than the second. That is, $M^*(n+1)$ is the smallest number to satisfy:

$$P_s\cdot q(n+1)\cdot Q(n,M) + (1-P_s\cdot q(n+1))Q(n+1,M)$$
$$< (1-q(n+1))P_p\cdot Q(n+1,M)$$

For all M, such that $M \leq M^*(n+1)$ we can write:

$$Q(n+1,M) = [1-P_p \cdot q(n+1)]^M,$$

and since $M^*(n+1)$ is assumed to be less than or equal to $M^*(n)$, we have also $M \leq M^*(n)$ and hence

$$Q(n,M) = [1-P_p \cdot q(n)]^M .$$

Thus, by substituting the last two equations into the inequality above, we find that $M^*(n+1)$ must be the first integer to satisfy

$$\left(\frac{1-P_p \cdot q(n+1)}{1-P_p \cdot q(n)}\right)^M > \frac{P_s}{P_s - P_p},$$

or:

$$M^*(n+1) = 1 + \left\lfloor \frac{\ln(1 - \frac{P_p}{P_s})}{\ln(\frac{1-P_p \cdot q(n)}{1-P_p \cdot q(n+1)})} \right\rfloor \qquad \text{(III.10)}$$

(notice that when $n = 0$, we have $q(0) = 1$, and we have exactly the expression for $M^*(1)$ discovered in Chapter II, Eq. (II.6)).

We can now proceed to carry out the induction step to complete the proof of the theorem. We have already proven the theorem for $N = 2$. Let us now show that if the theorem is valid for some N, then it is valid for N+1. The assumption that it is valid for N implies, among other things, that

$$r(1) \leq r(2) \leq \ldots \leq r(N)$$

and that

$$M^*(1) \geq M^*(2) \geq \ldots \geq M^*(N)$$

Now, it was already proven that an equivalent condition to $M^*(N+1) \leq M^*(N)$ is (see Eq. (III.8)):

$$M^*(N) \geq \frac{\ln\left(\frac{P_s}{P_s - P_p}\right)}{\ln\left(\frac{1 - P_p \cdot q(N+1)}{1 - P_p \cdot q(N)}\right)} \quad . \qquad \text{(III.11)}$$

On the other hand, the assumption $M^*(N) \leq M^*(N-1)$ (which is included in the induction hypothesis) implies (by the discussion above, which led to Eq. (III.10)) that:

$$M^*(N) = 1 + \left[\frac{\ln\left(1 - \frac{P_p}{P_s}\right)}{\ln\left(\frac{1 - P_p \cdot q(N-1)}{1 - P_p \cdot q(N)}\right)}\right] > \frac{\ln\left(1 - \frac{P_p}{P_s}\right)}{\ln\left(\frac{1 - P_p \cdot q(N-1)}{1 - P_p \cdot q(N)}\right)} \,.$$

If now we replace $M^*(N)$ by a smaller quantity, given in the right-hand side of this last inequality, we obviously retain the sufficiency of condition (III.11) for $M^*(N+1) \leq M^*(N)$ (but obviously not the necessity). We have

$$\frac{\ln\left(1 - \frac{P_p}{P_s}\right)}{\ln\left(\frac{1 - P_p \cdot q(N-1)}{1 - P_p \cdot q(N)}\right)} \geq \frac{\ln\left(\frac{P_s}{P_s - P_p}\right)}{\ln\left(\frac{1 - P_p \cdot q(N+1)}{1 - P_p \cdot q(N)}\right)} \,,$$

which is equivalent to:

$$\frac{1 - P_p \cdot q(N)}{1 - P_p \cdot q(N-1)} \leq \frac{1 - P_p \cdot q(N+1)}{1 - P_p \cdot q(N)}$$

or to

$$r(N) \leq r(N+1),$$

as a sufficient condition for $M^*(N+1) \leq M^*(N)$. The proof of Theorem 1 is thus complete.

The operational significance of this result is very transparent. As implied by the foregoing analysis, for every value of N_s, the number $M^*(N_s)+1$ is the minimum number of missiles the attacker should have in his stockpile in order for it to be worthwhile spending the first missile (at least) on a secondary target. The MMPR property indicates that the marginal growth of the miss probability goes up as the number of secondary targets increases. Thus it becomes more pressing for the attacker to reduce the number of secondary targets as this number increases. The non-increasing property of $M^*(n)$ is just an alternative way in which this last fact is viewed. It means that if the attacker has a given number of missiles, then as there are more secondary targets it becomes more compelling to aim (at least one) missile at the secondary targets. In other words, the minimum number of missiles he must have in order for it to be justifiable to spend at least one on secondary targets, decreases as the number of secondary targets increases.

We finish this section by presenting a numerical example of non-increasing $M^*(n)$ sequence. In designing this example we used the equivalence of the requirement $r(n+1) \geq r(n)$ and the requirement:

$$q(n+1) \leq \frac{1-P_p \cdot q(n-1) - (1-P_p \cdot q(n))^2}{P_p \cdot (1-P_p \cdot q(n-1))}$$

We took $P_p = 0.5$, $P_s = 0.9$, and designed the following case:

N_s	1	2	3	4	5
$q(N_s)$	0.95	0.85	0.7	0.7	0.268
$M^*(N_s)$	17	9	7	6	6

so that $M^*(n)$ is non-increasing on the set $\{n: 1 \leq n \leq 5\}$.

4. Increasing M*--Sequences--Solutions

Although situations in which the survival function possesses the MMPR property quite possibly exist, it was found by thorough explorations made by the author, that for most survival functions of practical relevance (see Section III.B), this property does not exist and the M*-sequences are actually (strictly) monotone increasing, i.e., $M^*(n+1) > M^*(n)$ for all n. Calculation of $M^*(n)$ (for all n) in this case is much more complicated, and we now turn to solve this general case. For the convenience of the following mathematical development we define the following function:

$$R(n,k) = Q(n, M^*(n)+k)$$

where we now use n (not N_s) to denote the number of secondary targets. The value $R(n,k)$ is the probability of not hitting the primary target, when n defense units defend it and the attackers stockpile contains k missiles more than the maximum

level $(M^*(n))$ that would require spending the whole stockpile on primary targets. From Eq. (II.10) we have:

$$R(1,k) = A_1\cdot(1-P_p)^k + B_{11}(1-P_s\cdot q(1))^k \qquad \text{(III.11)}$$

where:

$$A_1 = \frac{(1-P_p)^{M^*(1)}\cdot P_s\cdot q(1)}{P_s\cdot q(1) - P_p} \qquad \text{(III.11a)}$$

and

$$B_{11} = (1-q(1)\cdot P_p)^{M^*(1)} - \frac{(1-P_p)^{M^*(1)}\cdot P_s\cdot q(1)}{P_s\cdot q(1)-P_p}$$

$$= (1-q(1)\cdot P_p)^{M^*(1)} - A_1 \qquad \text{(III.11b)}$$

We now prove the following useful lemma:

<u>Lemma</u>: Let $q(n)$ be a strictly decreasing function of n. Then, for any n,k, the function $R(n,k)$ can be expressed by:

$$R(n,k) = A_n\cdot(1-P_p)^k + \sum_{j=1}^{n} B_{n,j}\cdot[1-P_s\cdot q(j)]^k \qquad \text{(III.12)}$$

where A_n and $\{B_{n,j}: j = 1,2,\ldots,n\}$ are constants which can be calculated recursively and are functions of the parameters of the problem only (that is, of P_p, P_s and $\{q(n)\}$).

Proof: We prove the lemma by induction. Its validity for $n = 1$ is already proven (see Eqs. (III.11), (III.11a),(III.11b) above). Assuming that it is true for all values of n up to some N, we show that it is valid for $n = N+1$. To do that, we consider the state $(N+1, M^*(N+1)+k)$. The first shot in this state should obviously be anti-secondary. We can calculate $R(N+1,k)$ by conditioning on the number of missiles which would be expended before achieving the first successful anti-secondary shot. If the first missile to hit a secondary target is the jth one, where $j \leq k$, then the attacker will be in state $(N, M^*(N+1)+k-j)$ from which the probability of not destroying the primary target is given by:

$$R(N, M^*(N+1)+k-j-M^*(N)) \quad = \quad Q(N, M^*(N+1)+k-j)$$

since $k-j \geq 0$ in this case and $M^*(n+1) > M^*(n)$, we deduce that $M^*(n+1)+k-j-M^*(n) > 0$ and so we can express this function using the induction hypothesis for $n = N$.

If, on the other hand, the first k missiles fail to destroy even a single secondary target, then the offender moves to state $(N+1, M^*(N+1))$ in which the probability of eventually missing the primary is simply $[1-P_p \cdot q(N+1)]^{M^*(N+1)}$.

Now, the probability of first hitting a secondary target on the jth missile is $(1-P_s \cdot q(N+1))^{j-1} \cdot P_s \cdot q(N+1)$, and the probability of not hitting any secondary target with the first k missiles is $(1-P_s \cdot q(N+1))^k$. Hence we can write

$$R(N+1,k) = Q(N+1,M^*(N+1)+k)$$

$$= \sum_{j=1}^{k} (1-q(N+1)\cdot P_s)^{j-1}\cdot q(N+1)P_s\cdot R(N,\Delta M^*(N)+k-j)$$

$$+ (1-q(N+1)\cdot P_s)^k\cdot(1-q(N+1)\cdot P_p)^{M^*(N+1)}$$

where we've substituted $\Delta M^*(N)$ for $M^*(N+1) - M^*(N)$.

We now proceed in developing the last expression by substituting for $R(N,\Delta M^*(N)+k-j)$ the hypothesized expression, which is taken valid for N by the induction assumption:

$$R(N+1,k) = \sum_{j=1}^{k} (1-q(N+1)\cdot P_s)^{j-1}\cdot q(N+1)\cdot P_s[A_N(1-P_p)^{\Delta M^*(N)+k-j}$$

$$+ \sum_{\ell=1}^{N} B_{N,\ell}(1-P_s\cdot q(\ell))^{\Delta M^*(N)+k-j}]$$

$$+ [1-q(N+1)\cdot P_s]^k\cdot(1-q(N+1)\cdot P_p)^{M^*(N+1)} \qquad \text{(III.13)}$$

A careful inspection of this rather awkward expression shows that it can be written as:

$$R(N+1,k) = R_0\cdot(1-P_p)^k + \sum_{\ell=1}^{N+1} R_{1,\ell}\cdot(1-q(\ell)\cdot P_s)^k \qquad \text{(III.13a)}$$

where R_0 and $R_{1,\ell}$, $\ell = 1,2,\ldots,n+1$ are complicated expressions manipulated below:

$$R_0 = \left[q(N+1)\cdot P_s(1-P_p)^{\Delta M^*(N)-1}\cdot A_N \right] \cdot \sum_{j=1}^{k} \left[\frac{1-q(N+1)\cdot P_s}{1-P_p} \right]^{j-1}$$

$$= q(N+1)\cdot P_s(1-P_p)^{\Delta M^*(N)-1}\cdot A_N \cdot \frac{1-\left(\frac{1-q(N+1)\cdot P_s}{1-P_p}\right)^k}{1-\frac{1-q(N+1)\cdot P_s}{1-P_p}}$$

$$= \frac{q(N+1)\cdot P_s(1-P_p)^{\Delta M^*(N)}}{q(N+1)\cdot P_s - P_p}\cdot A_N \cdot \left[1-\left(\frac{1-q(N+1)\cdot P_s}{1-P_p}\right)^k\right] .$$

For $\ell = 1,2 \ldots, n$, we have after similar manipulation and by using the assumed strict decreasing nature of $q(N_s)$:

$$R_{1,\ell} = \frac{q(N+1)\cdot[1-P_s\cdot q(\ell)]^{\Delta M^*(N)}}{q(N+1) - q(\ell)}\cdot B_{N,\ell}$$

$$\cdot \left[1-\left(\frac{1-q(N+1)\cdot P_s}{1-q(\ell)\cdot P_s}\right)^k\right] \qquad \text{(III.13b)}$$

and finally:

$$R_{1,N+1} = (1-q(N+1)\cdot P_p)^{M^*(N+1)}$$

Now substituting the expressions for R_0, $R_{1,\ell}$ ($\ell = 1,2,\ldots,n+1$) above into Eq. (III.13a), rearranging and collect[illegible] terms, we get

$$R(n+1,k) = \left[\frac{q(N+1)\cdot P_s(1-P_p)^{\Delta M^*(N)}}{q(N+1)\cdot P_s - P_p} \cdot A_N \right] \cdot (1-P_p)^k$$

$$+ \sum_{\ell=1}^{N} \left[\frac{q(N+1)(1-P_s q(\ell))^{\Delta M^*(N)}}{q(N+1) - q(\ell)} \cdot B_{N,\ell} \right] \cdot (1-q(\ell)\cdot P_s)^k$$

$$+ \Bigg[(1-q(N+1)\cdot P_s)^k [(1-q(N+1)\cdot P_p)^{M^*(N+1)}$$

$$- \frac{q(N+1)\cdot P_s(1-P_p)^{\Delta M^*(N)} \cdot A_N}{q(N+1)\cdot P_s - P_p}$$

$$- \sum_{\ell=1}^{N} \frac{q(N+1)\cdot[1-P_s\cdot q(\ell)]^{\Delta M^*(N)} \cdot B_{N,\ell}}{q(N+1) - q(\ell)} \Bigg]$$

The proof the lemma is now complete since the last expression has exactly the required form:

$$R(N+1,k) = A_{N+1}\cdot(1-P_p)^k + \sum_{\ell=1}^{N+1} B_{N+1,\ell}\cdot(1-q(\ell)\cdot P_s)^k \qquad \text{(III.14)}$$

$$= Q(N+1,M^*(N+1)+k)$$

The constants A_{N+1} and $B_{N+1,\ell}$, $\ell = 1,2,\ldots,n+1$ can be calculated from A_N and $B_{N,\ell}$, $\ell = 1,2,\ldots,N$ as follows:

$$A_{N+1} = \frac{q(N+1)\cdot P_s(1-P_p)^{\Delta M^*(N)}}{q(N+1)P_s - P_p} \cdot A_N \qquad \text{(III.14a)}$$

$$B_{N+1,\ell} = \begin{cases} \dfrac{q(N+1)\cdot(1-P_s\cdot q(\ell))^{\Delta M^*(N)}}{q(N+1) - q(\ell)}\cdot B_{N,\ell} & \text{for } \ell \leq N \\[2ex] (1-q(N+1)\cdot P_p)^{M^*(N+1)} - A_{N+1} - \sum_{j=1}^{N} B_{N+1,j} & \text{for } \ell = N+1. \end{cases} \tag{III.14b}$$

Notice that the constants A_1 and B_{11} are known (Eqs. III.11a-11b). Hence, if the function $M^*(n)$ was found, we could use it, along with the data of the problem, to calculate A_n and $B_{n,\ell}$ for all n and ℓ (using formulae (III.14a-14b) and thus to have a very convenient expression of the miss probabilities $Q(n,M)$ for $M > M^*(n)$. The function $M^*(n)$ itself can be calculated recursively as we now show:

Suppose $M^*(1), M^*(2), \ldots, M^*(n)$ are already known. Let $N_s = n+1$. We wish to find $M^*(n+1)$. Returning to the functional Equation (III.1), we write for a state $(n+1,M+1)$ where $M \geq M^*(n)$:

$$Q(n+1,M+1) = \operatorname{Min}\{P_s\cdot q(n+1)\cdot Q(n,M)+(1-P_s\cdot q(n+1))\cdot Q(n+1,M), \\ (1-q(n+1)\cdot P_p)\cdot Q(n+1,M)\}$$

By its definition, $M^*(n+1)$ should be the least value of M such that the minimum is attained at the first term in brackets. This is so because among all states of the form $(n+1,M)$, the first M that makes it optimal to launch a first missile against a secondary target, is (by definition) $M = M^*(n+1) + 1$. The first term in brackets corresponds to the decision to launch

an anti-secondary missile. Hence, $M^*(n+1)$ should be the least M such that:

$$P_s \cdot q(n+1) \cdot Q(n,M) + (1-P_s \cdot q(n+1)) \cdot Q(n+1,M)$$

$$< \quad (1-q(n+1) \cdot P_p) \cdot Q(n+1,M). \qquad \text{(III.15)}$$

Now, we have to imagine that we check the numbers $M^*(n)+1$, $M^*(n)+2$, and so on, to see which is the least one to satisfy this inequality. For any M such that $M = M^*(n)+k \leq M^*(n+1)$ we have:

$$Q(n+1,M) \quad = \quad (1-P_p \cdot q(n+1))^{M^*(n)+k} \qquad \text{(III.15a)}$$

whereas for states (n,M) (where $M > M^*(n)$) we proved that:

$$Q(n,M) \quad = \quad R(n,M-M^*(n))$$

$$= \quad A_n(1-P_p)^{M-M^*(n)} + \sum_{j=1}^{n} B_{n,j}(1-P_s \cdot q(j))^{M-M^*(n)} \qquad \text{(III.15b)}$$

Substituting now Eq. (III.15a) and Eq. (III.15b) in Eq. (III.15) and simplifying, we conclude that $M^*(n+1)$ is the least number M to satisfy the following inequality:

$$A_n(1-P_p)^{M-M^*(n)} + \sum_{j=1}^{n} B_{n,j}(1-P_s \cdot q(1))^{M-M^*(n)}$$

$$< \quad (1 - \frac{P_p}{P_s})(1 - P_p \cdot q(n+1))^M. \qquad \text{(III.16)}$$

The results at which we arrived here clearly provide a complete solution to the optimal allocation problem with MPH

criterion, for which it is known before that the function $M^*(n)$ is (strictly) increasing. For each n, it can be decided whether $M^*(n+1) > M^*(n)$ or not by checking whether inequality (III.7) is satisfied or not. If it is known that $M^*(n+1) > M^*(n)$, then $M^*(n+1)$ can be found from $M^*(1), M^*(2), \ldots, M^*(n)$ and the constants A_n, $\{B_{n,j}: j = 1,2,\ldots,n\}$ by solving for the least M to satisfy inequality (III.16). Then A_{n+1} and $\{B_{n+1,j}: j = 1,2,\ldots,n+1\}$ can be calculated, using Eqs. (III.14a) and (III.14b), and the whole process is then repeated, to find $M^*(n+2)$ (assuming, again, $M^*(n+2) > M^*(n+1)$).

In Fig. III.1 we present a block diagram of the computational algorithm which solves the allocation problem with the MPH criterion for strictly increasing M*-sequences. The diagram assumes that the M*-sequence is already known to be monotone increasing. The confirmation of this fact is very simple, as was explained before, and to include that we only had to add a special loop that checks, for every n, whether $M^*(n)$ still increases. However, since most interesting survival functions are known already (as mentioned before) to have the strict monotonicity property, we have preferred to eliminate this loop to avoid an obscure representation of the algorithm.

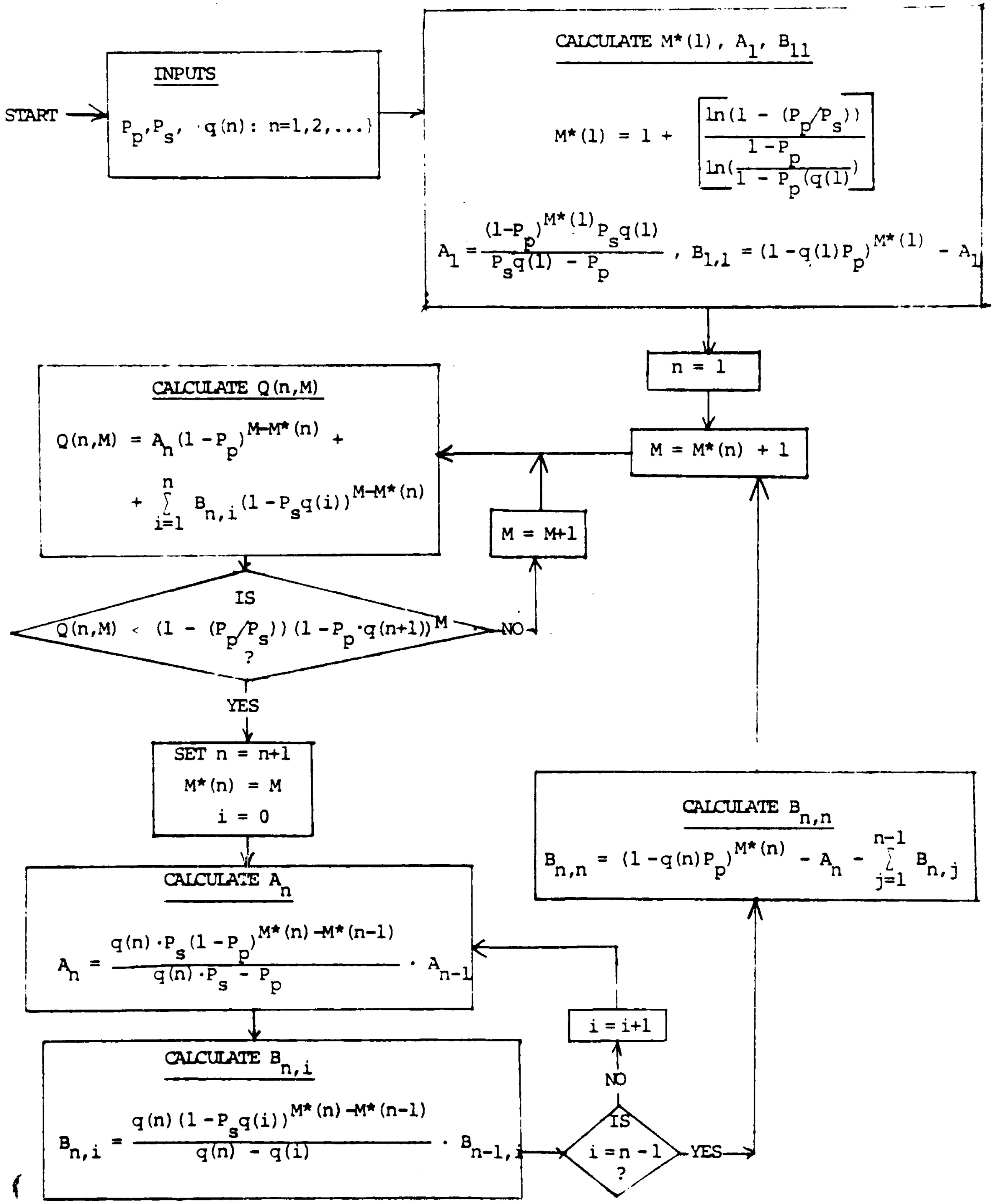

FIGURE III.1: Flow Chart of Computational Program to Solve Optimal Allocation Problem (MPH Criterion)

D. OPTIMAL ALLOCATION--MAX. EXPECTED NUMBER OF PENETRATORS (MENP) CRITERION

1. Formulation

The Max. Expected Number of Penetrators (MENP) criterion may best fit a situation in which the primary target is not a single point target, but rather a complex consisting of many point targets. This criterion fits also "area targets"--targets which have continuously distributed value over a large area (See Eckler & Burr, [1]). An airbase is a good example of such a target. In this case, a very convenient and sensible measure of effectiveness is the expected number of anti-primary missiles which penetrate the defense.

We denote by $E(N_s,M)$ the maximum expected number of AP missiles succeeding to penetrate the defense. When at the initial state the attacker has M launch opportunities and the defender has N_s secondary targets, the functional equation for $E(N_s,M)$ is:

$$E(N_s,M) = \text{Max}\{P_s \cdot q(N_s) \cdot E(N_s-1,M-1) + (1-P_s \cdot q(N_s)) \cdot E(N_s,M-1), \quad P_p \cdot q(N_s) + E(N_s,M-1)\} \tag{III.17}$$

It was explained in Chapter II that we can take $P_p = 1$ without loss of generality, since the optimal policies do not depend on P_p, and the value of the objective function at the optimum is simply proportional to P_p. The first term inside the brackets of Eq. (III.17) expresses, as usual, the value of the objective when the first decision of the attacker is AS, and he then proceeds optimally.

We observe that there exist, as in the allocation problem with the MPH criterion, optimal policies such that no switch from an AP decision to an AS decision is allowed. If $q(N_s)$ is strictly decreasing, we can say more, namely, that no optimal policy exists which allows such a switch. This property was proved for problems with the MPH criterion. For the MENP criterion, its existence is even more intuitively obvious. (The mathematical proof is immediate and we omit it here.)

Thus we have the notion of the M*-sequence as before. All we need to do to determine the optimal policy is to calculate the M*-sequence (which for a given set of parameters and a given survival function may differ of course from the M*-sequence resulting if we adopt the MPH criterion).

2. Non-increasing M*-Sequence--Relation to Concavity of the Survival Function

First notice that $M^*(1)$ is exactly M^* that was calculated in Chapter II and is given by Eq. (II.14):

$$M^*(1) = 1 + \left[\frac{1}{P_s(1-q(1))}\right] . \tag{III.18}$$

Suppose now that for some n we have $M^*(n) < M^*(n+1)$. This was shown earlier (see proof of Theorem 1) to be equivalent to

$$D^*(n+1, M^*(n)+1) = AP \tag{III.19}$$

and this property is further equivalent to:

$$q(n+1)+E(n+1,M^*(n)) > P_s \cdot q(n+1) \cdot E(n,M^*(n)) + (1-P_s \cdot q(n+1)) \cdot E(n+1,M^*(n)) . \quad \text{(III.20)}$$

Eq. (III.20) is implied by interpreting Eq. (III.19) as having the second term in the functional equation (III.17) greater than the first (where N_s and M are replaced, respectively, by n+1 and M*(n)+1 in that equation). We can simplify Eq. (III.20) by rearranging terms and have:

$$E(n,M^*(n)) - E(n+1,M^*(n)) < \frac{1}{P_s} . \quad \text{(III.21)}$$

In state (n,M*(n)) the optimal policy dictates to use AP missiles only. The probability of penetration of each missile is q(n). Hence we can write:

$$E(n,M^*(n)) = q(n) \cdot M^*(n) .$$

Since $M^*(n+1) > M^*(n)$ we have the same situation in state (n+1,M*(n)), that is, that only AP missiles are used at the optimal policy, thus:

$$E(n+1,M^*(n)) = q(n+1) \cdot M^*(n) .$$

Substituting the last two equalities in Eq. (III.21) and simplifying we reach the following inequality:

$$M^*(n) < \frac{1}{P_s(q(n)-q(n+1))} . \quad \text{(III.22)}$$

Thus we showed that $M^*(n) < M^*(n+1)$ is equivalent to Eq. (III.22). For our needs we prefer to emphasize the

equivalence of the opposite statements, namely, that $M^*(n) > M^*(n+1)$ is equivalent to

$$M^*(n) \geq \frac{1}{P_s(q(n)-q(n+1))} \quad . \qquad \text{(III.23)}$$

If $M^*(n) \geq M^*(n+1)$ we can quite simply calculate $M^*(n+1)$ in the following way. We write the functional equation for state $(n+1,M+1)$:

$$E(n+1,M+1) = \text{Max}\{P_s \cdot q(n+1) \cdot E(n,M) + (1-P_s \cdot q(n+1)) \cdot E(n+1,M),$$
$$q(n+1) + E(n+1,M)\}$$

By definition, $M^*(n+1)$ is the smallest M for which the first term is greater than the second. That is, $M^*(n+1)$ is the smallest M which satisfies:

$$P_s[E(n,M) - E(n+1,M)] > 1.$$

For $M < M^*(n+1)$ we clearly have $E(n+1,M) = q(n+1) \cdot M$, and since we assume $M^*(n+1) \leq M^*(n)$, then $M < M^*(n+1)$ implies also $M < M^*(n)$ so that we have also $E(n,M) = q(n) \cdot M$. Thus, $M^*(n+1)$ is the smallest M to satisfy:

$$P_s \cdot M[q(n) - q(n+1)] > 1,$$

or

$$M^*(n+1) = 1 + \left[\frac{1}{P_s(q(n)-q(n+1))}\right] \quad . \qquad \text{(III.24)}$$

We now present a theorem which gives a sufficient condition for the function $M^*(n)$ to be non-increasing.

We will say that the function $q(n)$ defined on a set $\{n: 1 \le n \le N\}$ is "concave" on the set iff:

$$q(n-1) - q(n) \le q(n) - q(n+1)$$

for all $n \le N-1$. This is analogous to the property of concave functions defined on an interval $[a,b]$ of the real line, where the formal definition of concavity is that for all $\lambda \in [0,1]$, the function (call it f) satisfies:

$$f(\lambda a + (1-\lambda)b) \ge \lambda \cdot f(a) + (1-\lambda)f(b) .$$

<u>Theorem 2</u>: A sufficient condition for the function $M^*(n)$ to be non-increasing on the region $1 \le n \le N$ is that the survival function $q(n)$ be (strictly) concave on the region $1 \le n \le N$.

The theorem is of course analogous to Theorem 1, where a sufficient condition for the non-increasing property of $M^*(n)$ was given for the optimal allocation problem with the MPH criterion. We will later discuss the similarities in the operational interpretations of the two sufficient conditions, given in the two theorems. Notice that in both conditions, the parameter P_s (the hit probability of a secondary target, given survival of the missile), does not play any role. We now prove the above theorem.

Proof: We will use induction. First we show that if the survival function is concave on the set $\{0,1,2\}$, then $M^*(2) \leq M^*(1)$. The concavity means that

$$q(0) - q(1) \leq q(1) - q(2). \quad (q(0) = 1 \text{ by assumption})$$

We have shown (Eq. III.18) that:

$$M^*(1) = 1 + \left[\frac{1}{P_s \cdot (1-q(1))}\right] > \frac{1}{P_s(1-q(1))} \qquad \text{(III.25)}$$

Eq. (III.23) states that a necessary and sufficient condition for $M^*(2) \leq M^*(1)$ is that

$$M^*(1) \geq \frac{1}{P_s(q(1)-q(2))} \qquad \text{(II.25a)}$$

From (III.25) it is obvious that a sufficient condition for Eq. (III.25a) to be valid is

$$\frac{1}{P_s(1-q(1))} \geq \frac{1}{P_s(q(1)-q(2))},$$

or

$$q(0) - q(1) = 1-q(1) \leq q(1) - q(2),$$

so that the theorem is proven for N = 2. Now suppose it is true for some N. We show that it is true for N+1 also. The induction hypothesis thus includes the assumption $M^*(n) \geq M^*(n-1)$ for all $n \leq N$. We must show that a sufficient condition for $M^*(N+1) \leq M^*(N)$ is that:

$$q(N-1) - q(N) \leq q(N) - q(N+1).$$

To show this, notice that the assumption $M^*(N) \leq M^*(N-1)$ implies, as was explained by the argument which led to Eq. (III.24), that

$$M^*(N) = 1 + \left[\frac{1}{P_s(q(N-1)-q(N))}\right] .$$

On the other hand, Eq. (III.23) shows that a condition equivalent for $M^*(N+1) \leq M^*(N)$ is:

$$M^*(N) \geq \frac{1}{P_s(q(N)-q(N+1))} .$$

The last two expressions together lead to the sufficiency of the concavity condition

$$q(N-1) - q(N) \leq q(N) - q(N+1)$$

for the relation

$$M^*(N+1) \leq M^*(N).$$

The proof of the theorem is thus complete!

The operational interpretation of the concavity condition is quite obvious. The meaning of the concavity property is quite analogous to the meaning of the MMPR property discussed with relation to the problem with the MPH criterion. The analogy lies in the fact that both conditions reflect, each in its own way, a phenomenon of growing marginal effects. Both conditions indicate that the marginal effect induced by adding one secondary target is becoming worse as more secondary

targets are introduced. Here in the case of the MEP criterion, the marginal effect is simply the decrease in survivability, and concavity means that this decrease gets sharper as N_s increases.

It is noteworthy that concave survival functions are not so much expected to occur in the real world. The most common function, $q(N_s) = q_0^{N_s}$ (corresponding to independent operations of the secondary targets) is convex. Other forms of the survival function with realistic appeal are also non-concave. This does not prove, however, that $M^*(N_s)$ is monotone increasing, since the concavity of $q(N_s)$ was shown to be sufficient only.

One enlightening example can be given for a concave survival function. Imagine a group of secondary targets, located around the primary target, such that each target is responsible for intercepting missiles arriving from an angular section of 120° (Fig. III.2). This example shows that the survivability should not be very much different if there are one or two secondary targets, because in both cases the attacker is able to find an undefended direction of penetration. If, however, we have $N_s = 3$, then the survivability drops sharply since the defender may then locate his units so as to leave no undefended direction of arrival.

3. Increasing M*-Sequences--Solutions

We proceed by presenting the computational procedure for solving the allocation problem when the function $M^*(n)$ is

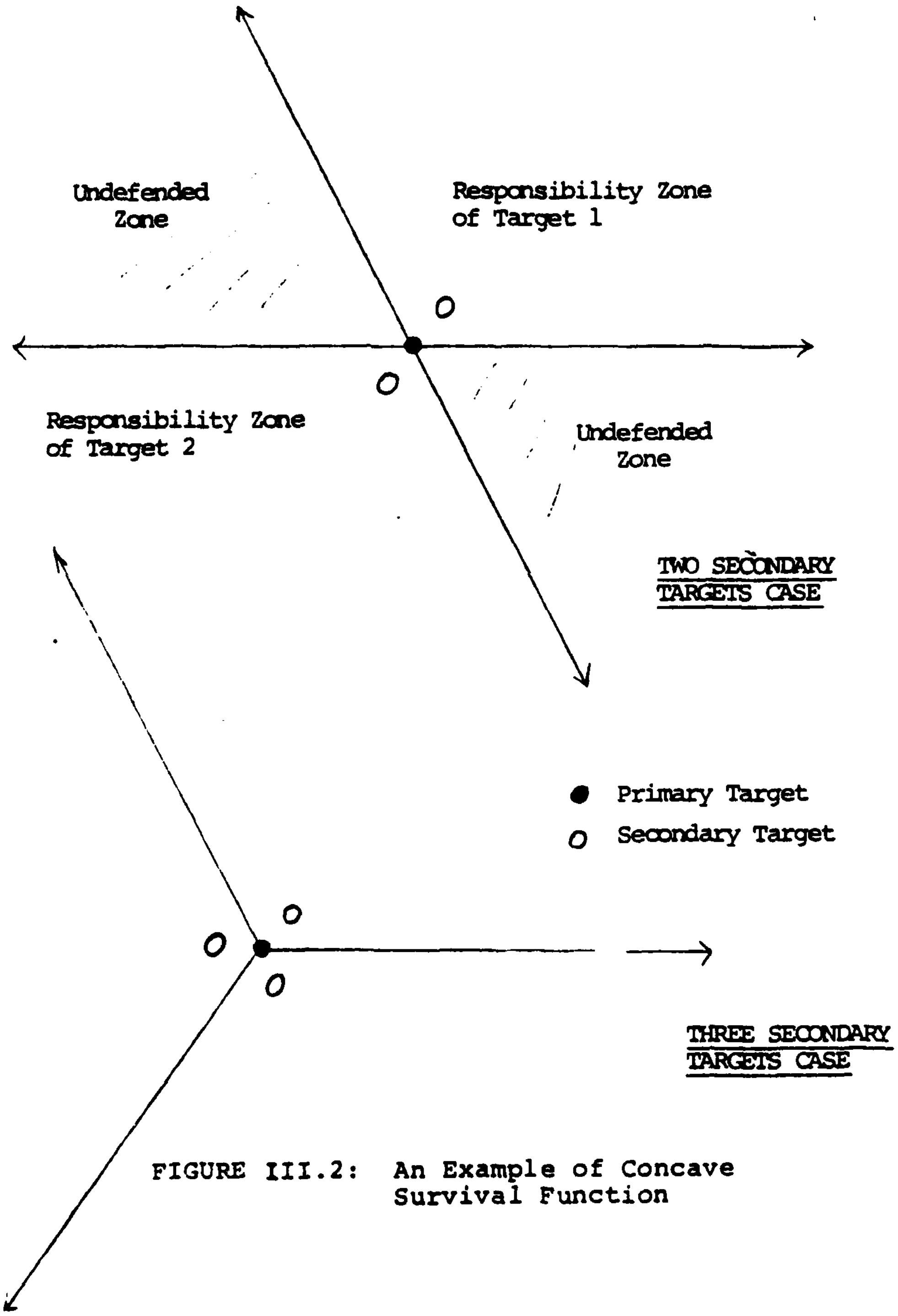

FIGURE III.2: An Example of Concave Survival Function

a strictly monotone increasing function. For all states (n,M) such that $M \leq M^*(n)$, it is implied by the definition of $M^*(n)$ that:

$$E(n,M) = M \cdot q(n).$$

In order to find the sequence $\{M^*(n): n = 1,2,\ldots\}$ we have to be able to express $E(n,M)$ for values of M such that $M > M^*(n)$. For $M^*(1)$ we have already the expression (Eq. (II.15)):

$$E(1,M) = M - \frac{1}{P_s \cdot q(1)} + \left(\frac{1}{P_s \cdot q(1)} - M^*(1)(1-q(1))\right) \cdot (1-P_s \cdot q(1))^{M-M^*(1)} \qquad \text{(III.26)}$$

where $q(1)$, $M^*(1)$ and $E(1,M)$ have replaced q, M^* and $E(M)$, respectively, in Eq. (II.15).

We now state and prove the following lemma:

<u>Lemma</u>: For a state (n,M) such that $M > M^*(n)$, the optimal expected number of penetrators can be expressed as follows:

$$E(n,M) = M - \frac{1}{P_s} \cdot \sum_{i=1}^{n} \frac{1}{q(i)} + \sum_{i=1}^{n} H_{n,i}(1-q(i) \cdot P_s)^{M-M^*(i)} \qquad \text{(III.27)}$$

where $\{H_{n,i}: n = 1,2,\ldots, i = 1,2,\ldots,n\}$ is a family of constants which can be calculated recursively (as will be shown in the course of the proof).

Proof: We prove the lemma by induction. First notice that Eq. (III.26) itself approves the validity of the lemma for $n = 1$, since $E(1,M)$ has exactly the form given in (III.27) with:

$$H_{1,1} = \frac{1}{P_s \cdot q(1)} - M^*(1) \cdot (1-q(1)) ,$$

where $M^*(1)$ is given by Eq. (III.18).

Let us therefore assume that $E(n,M)$ is given by Eq. (III.27). We calculate $E(n+1,M)$, for values of $M > M^*(n+1)$. To accomplish that, notice that we can write, for $M > M^*(n+1)$:

$$E(n+1,M) = \sum_{j=1}^{M-M^*(n+1)} (1-P_s \cdot q(n+1))^{j-1} \cdot P_s \cdot q(n+1) \cdot E(n,M-j)$$

$$+ (1-P_s \cdot q(n+1))^{M-M^*(n+1)} \cdot M^*(n+1) \cdot q(n+1) \qquad \text{(III.28)}$$

The argument which leads to Eq. (III.28) is the following: If the attacker achieves the first hit of a secondary target with the jth missile, where j is less than or equal to $M-M^*(n+1)$, then he is in state $(n,M-j)$, at which the expected number of penetrators is simply $E(n,M-j)$. The probability of a first hit to occur at the jth attempt is $(1-P_s \cdot q(n+1))^{j-1} \cdot P_s \cdot q(n+1)$. The first term accounts for all the cases in which a hit is achieved with one of the first $M-M^*(n+1)$ missiles. If all the first $M-M^*(n+1)$ missiles miss the secondary targets, an event which occurs with probability $(1-P_s \cdot q(n+1))^{M-M^*(n+1)}$, then the attacker finds himself in the state $(n+1,M^*(n+1))$, where by definition of $M^*(n+1)$, the

optimal policy is to launch at the primary target only. The optimal expected number of penetrators in that case is therefore $M^*(n+1) \cdot q(n+1)$.

Now notice, that since $M^*(n) < M^*(n+1)$, we have (if $j \leq M-M^*(n+1)$):

$$M-j \geq M^*(n+1) > M^*(n),$$

and thus $E(n,M-j)$, which appears in the first term on the right-hand side of Eq. (III.28), can be substituted by the expression assumed to be valid for it by the induction hypothesis (see Eq. (III.27) above). We thus have:

$$E(n+1,M) = \sum_{j=1}^{M-M^*(n+1)} (1-P_s \cdot q(n+1))^{j-1} \cdot P_s \cdot q(n+1)$$

$$\cdot \left[M-j-\frac{1}{P_s} \cdot \sum_{i=1}^{n} \frac{1}{q(i)} + \sum_{i=1}^{n} H_{n,i} (1-q(i) \cdot P_s)^{M-j-M^*(i)} \right]$$

$$+ (1-P_s \cdot q(n+1))^{M-M^*(n+1)} \cdot M^*(n+1) \cdot q(n+1)$$

(a very laborious algebraic work, aimed at simplifying this last expression, is now given without verbal explanation)

$$= \left[M - \frac{1}{P_s} \cdot \sum_{i=1}^{n} \frac{1}{q(i)} \right] \cdot \left[1 - (1-P_s \cdot q(n+1))^{M-M^*(n+1)} \right]$$

$$+ \sum_{i=1}^{n} H_{n,i} (1-q(i) \cdot P_s)^{M-M^*(i)}$$

$$\cdot \left[\frac{1 - \left(\frac{1-P_s \cdot q(n+1)}{1-P_s \cdot q(i)}\right)^{M-M^*(n+1)}}{1 - \left(\frac{1-P_s \cdot q(n+1)}{1-P_s \cdot q(i)}\right)} \right] \cdot \left(\frac{P_s \cdot q(n+1)}{1-q(i) \cdot P_s}\right)$$

$$+ \; (1-P_s \cdot q(n+1))^{M-M^*(n+1)} \cdot M^*(n+1) \cdot q(n+1)$$

$$- \; \frac{1-(M-M^*(n+1)+1)\,(1-P_s \cdot q(n+1))^{M-M^*(n+1)}}{P_s \cdot q(n+1)}$$

$$+ \; \frac{(M-M^*(n+1))\,(1-P_s \cdot q(n+1))^{M-M^*(n+1)+1}}{P_s \cdot q(n+1)}$$

$$= \; M - \frac{1}{P_s} \cdot \sum_{i=1}^{n+1} \frac{1}{q(i)} + \sum_{i=1}^{n} \frac{H_{n,i} \cdot q(n+1)}{q(n+1)-q(i)} \cdot (1-q(i) \cdot P_s)^{M-M^*(i)}$$

$$+ \Bigg[\frac{1}{P_s} \cdot \sum_{i=1}^{n} \frac{1}{q(i)} - M + M^*(N+1) \cdot q(n+1)$$

$$+ \; \frac{(M-M^*(n+1)+1)}{P_s \cdot q(n+1)} - \frac{(M-M^*(n+1))\,(1-P_s \cdot q(n+1))}{P_s \cdot q(n+1)}$$

$$- \; \sum_{i=1}^{n} H_{n,i} \cdot \frac{q(n+1)}{q(n+1)-q(i)} \cdot (1-q(i) \cdot P_s)^{M^*(n+1)-M^*(i)} \Bigg]$$

$$\cdot \, (1-P_s \cdot q(n+1))^{M-M^*(n+1)} \; .$$

This can be rewritten as follows:

$$E(n+1,M) \; = \; M - \frac{1}{P_s} \cdot \sum_{i=1}^{n+1} \frac{1}{q(i)} + \sum_{i=1}^{n+1} H_{n+1,i} (1-q(i) \cdot P_s)^{M-M^*(i)} \tag{III.29}$$

where the constants $H_{n+1,i}$ are defined as follows:

$$H_{n+1,i} = \begin{cases} \dfrac{H_{n,i}\cdot q(n+1)}{q(n+1)-q(i)} & \text{for } i \le n \quad \text{(III.30a)} \\[2ex] \dfrac{1}{P_s}\cdot\sum_{i=1}^{n+1}\dfrac{1}{q(i)} - M^*(n+1)(1-q(n+1)) & \\ \quad - \sum_{j=1}^{n} H_{n+1,j}(1-P_s\cdot q(j))^{M^*(n+1)-M^*(j)} & \text{(III.30b)} \\ & \text{for } i = n+1 \end{cases}$$

The lemma is now completely proven. Notice that the constants $H_{n,i}$ can very easily be calculated by the recursive scheme of computations given in Eqs. (III.30a) and (III.30b).

The calculations are initiated by first calculating $H_{1,1}$:

$$H_{1,1} = \frac{1}{P_s\cdot q(1)} - M^*(1)\cdot(1-q(1)). \qquad \text{(III.31)}$$

Notice also, that for calculating $H_{n+1,i}$ $(i = 1,2,\ldots,n+1)$, the value of $M^*(n+1)$ must be known already. We now show how $M^*(n+1)$ can also be calculated in a recursive manner. We assume that $M^*(n)$ is already known, and that $H_{n,i}$ $(i = 1,2,\ldots,n)$ are also known. We go back to the functional equation (III.17). We examine all states of the form $(n+1,M)$ where $M > M^*(n)$. By definition of $M^*(n+1)$, the least value of M such that $E(n+1,M)$ is attained by the first term under the max. operation in Eq. (III.17), should be $M^*(n+1)+1$. Writing this observation explicitly we find that $M^*(n+1)$ is the least value of M to satisfy the equation:

$$P_s \cdot [E(n,M) - E(n+1,M)] > 1 \ . \qquad \text{(III.32)}$$

If we start checking each M, from $M = M^*(n)+1$ and up for satisfying this equation, it is clear that we can substitute:

$$E(n+1,M) = M \cdot q(n+1)$$

$$E(n,M) = M - \frac{1}{P_s} \cdot \sum_{i=1}^{n} \frac{1}{q(i)} + \sum_{i=1}^{n} H_{n,i} (1-q(i) \cdot P_s)^{M-M^*(i)}$$

and thus have $M^*(n+1)$ defined as:

$$M^*(n+1) = \mathrm{Min}\{M > M^*(n): \ M(1-q(n+1)) - \frac{1}{P_s} \cdot \sum_{i=1}^{n} \frac{1}{q(i)}$$

$$+ \sum_{i=1}^{n} H_{n,i} (1-q(i) \cdot P_s)^{M-M^*(i)} > \frac{1}{P_s}\} \qquad \text{(III.32)}$$

We are now ready to summarize the computational procedure for solving the optimal allocation problem, where an arbitrary number of secondary targets are present, and for the MENP criterion. A flow chart of the algorithm is given in Fig. III.3. Notice that the two programs, the one with the MPH criterion, and this last one for the MENP criterion, are quite similar in flow logic. The difference, of course, is in the recursive schemes used to calculate the constants needed, and in the form of the test which is required to identify $M^*(n)$ for each n.

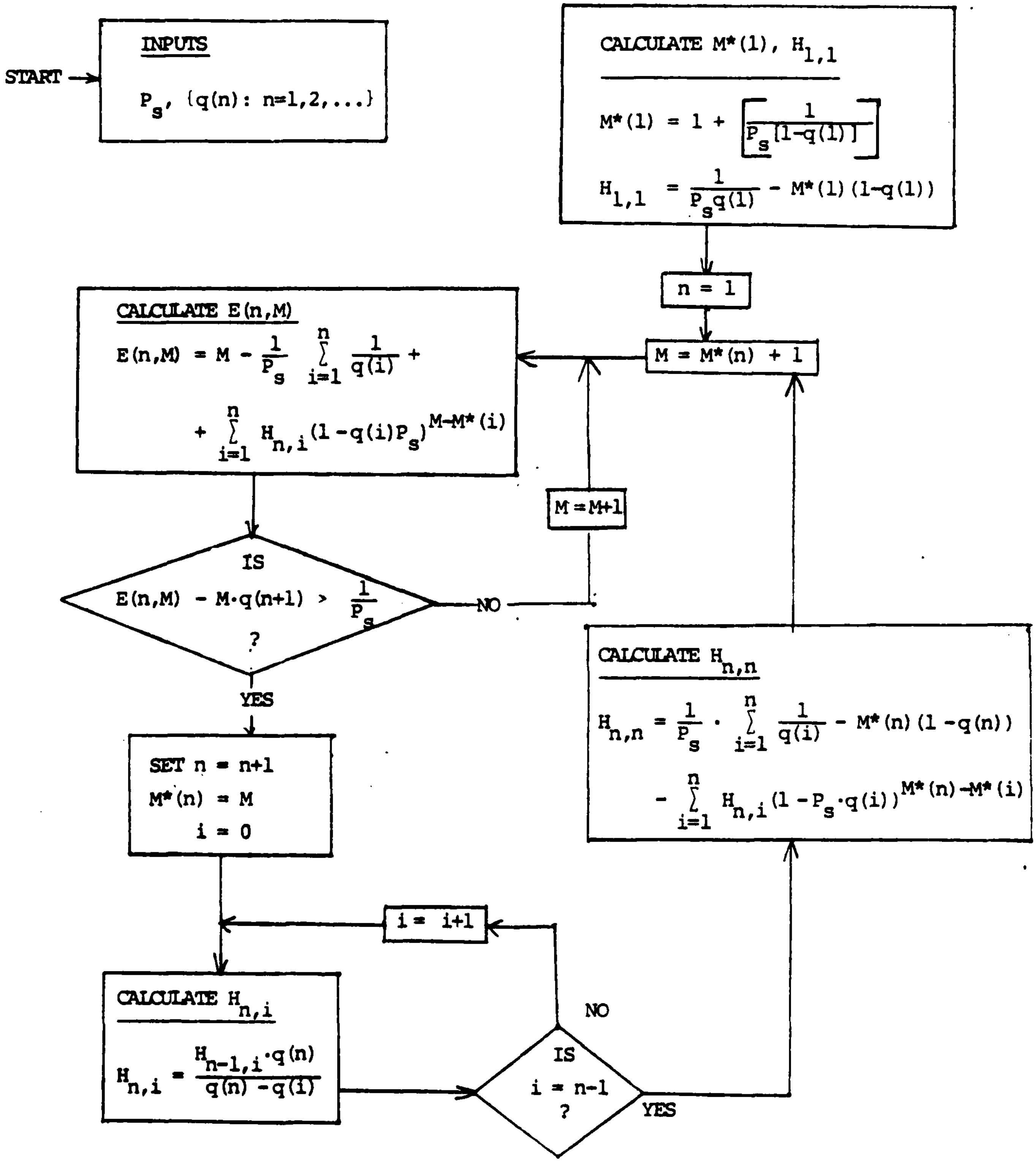

FIGURE III.3: Flow Chart of Computational Program to Solve Optimal Allocation Problem with MENP Criterion

E. OPTIMAL ALLOCATION--MINIMUM EXPECTED COST OF DESTROYING THE PRIMARY TARGET (MEC CRITERION)

1. Theory

This problem differs from the problems treated before in that the number of stages of the process is not determined *a priori*. It is decided at the beginning of the process that it will be allowed to go on until a number N_p of primary targets are destroyed.

This situation may occur when the operational 'worth' of the N_p primary targets is very high, and the number of missiles the attacker can spend on this mission is practically unlimited. In that case, an attacker can very well decide that he "won't stop" before the mission is fulfilled. The problem then remains, how to carry out the mission with minimum expected cost.

We assume that C_p and C_s are the unit-costs of an anti-primary and anti-secondary missile, respectively.* We characterize a 'state' by the pair (N_s,N_p) where, as usual, N_s is the number of secondary targets, and N_p is the number of primary targets.

As with the other two criteria, here also, no switch from anti-primary course to anti-secondary one can ever be an

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*One must keep in mind that estimating the costs C_p and C_s for actual implementation of the model we present here, may be a very hard task. Different estimates may be suggested, depending upon at what stage of the development the decision about the missiles mix is taken. This is, however, irrelevant to the model building process.

optimal move. Once the attacker starts attacking primary targets, he should stay on primary targets until the end of the process. In this model this is a very intuitive and transparent conclusion, so that it doesn't deserve any further mathematical justification.

When $N_s = 0$, then the optimal policy is trivial, since only primary targets exist. We can define:

$$N_s^*(N_p) = \text{Min}\{N_s: \; D^*(N_s+1,N_p) = AS\} ,$$

where D* is the optimal policy (a function of (N_s,N_p)). The number $N_p^*(N_p)$ has the following meaning: For a given number (N_p) of primary targets, the number $N_s^*(N_p)+1$ is the least number of secondary targets that should be present, in order for it to be optimal to deal with a secondary target first. Notice that we did not define $N_s^*(N_p)$ as:

$$N_s^*(N_p) = \text{Max}\{N_s: \; D^*(N_s,N_p) = AP\} ,$$

and that is because nothing in theory excludes the possibility that there exist values of N_s greater than $N_s^*(N_p)+1$, such that the optimal course in states (N_s,N_p), for those values, is again anti-primary. We define:

$$N_s^{**}(N_p) = \text{Min}\{N_s > N_s^*(N_p): \quad D^*(N_s+1,N_p) = AP\}$$

The significance of the functions $N_s^*(N_p)$ and $N_s^{**}(N_p)$ is explained in Fig. III.4. This figure depicts the general

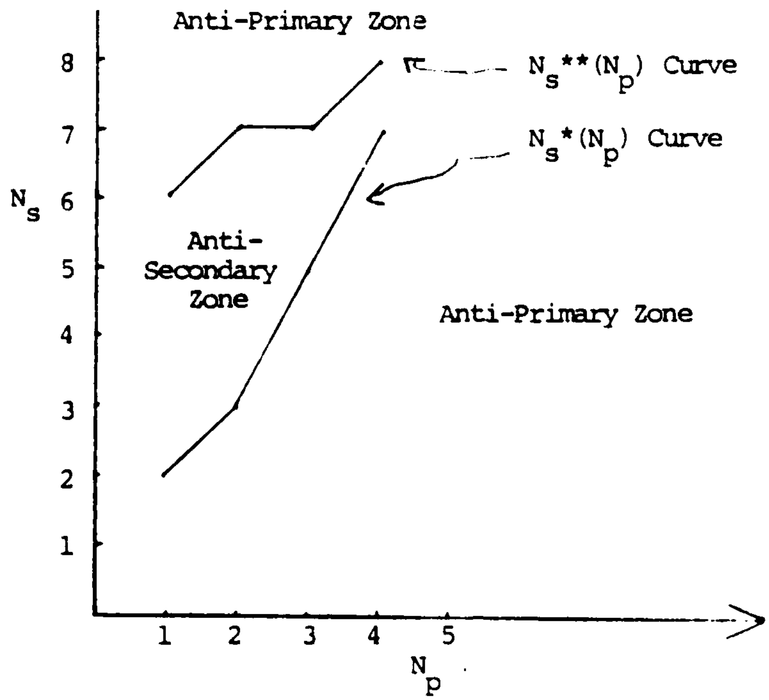

Figure III.4: The General Structure of Optimal Allocation Policy with MEC Criterion.

structure of optimal policies with MEC criterion. The operational interpretation of the existence of curves $N_s^*(N_p)$ and $N_s^{**}(N_p)$ separating different decision zones in the (N_p, N_s) plane is quite clear. For a given N_p, there are the following different possibilities:

a) It may be that N_s is small enough, so that the probability of survival will be high enough to make it worthwhile just to ignore the presence of the secondary targets, and go directly against the primary ones, accepting the small rate of attrition. The number $N_s^*(N_p)$ is just the function which indicates what numbers of secondary targets should be regarded as 'small' within that context. (The answer is: 'small' is less than or equal to $N_s^*(N_p)$).

b) The number N_s may be so large, so that it would consume too many anti-secondary missiles to make any significant reduction in the attrition rate. In fact, so many anti-secondary missiles may be required, that it is preferable just to give up attacking secondary targets, and use only anti-primary missiles, accepting the high rate of attrition. Again, the question is, what values of N_s are 'large' so that this argument is valid. The answer: Those values which are greater than or equal to $N_s^{**}(N_p)$.

c) For the given N_p, all the values of N_s, such that $N_s^{*}(N_p) < N_s \le N_s^{**}(N_p)$ are neither too small nor too large, so that it is preferable to start shooting at secondary targets for the benefit of later reduced rate of attrition.

In principle, it is possible to have more values of N_s (for a given N_p) which are points of transition from anti-primary optimal decisions to anti-secondary optimal decision. For example, we may have a value $N_s = N_s^{***}(N_p)$ such that:

$$N_s^{***}(N_p) = \mathrm{Min}\{N_s > N_s^{**}(N_p)+1 : D^*(N_s+1, N_p) = AS\} .$$

The existence of a finite $N_s^{***}(N_p)$ obviously depends on the survival function $q(N_s)$. It was found, however, that for all real, interesting and sensible survival functions (which are explored later), finite values of $N_s^{***}(N_p)$ never exist. Thus, in all the cases examined in this thesis, $N_s^{**}(N_p)$ has the property that:

$$D^*(N_s, N_p) = AP \quad \text{for all} \quad N_s > N_s^{**}(N_p).$$

We proceed by looking for a general analytic procedure of solving the allocation problem. Given a survival function $q(N_s)$ and the parameters P_p, P_s, we have to show how to find

the functions $N_s^*(N_p)$ and $N_s^{**}(N_p)$ and the expected cost function. Let $C_D(N_s,N_p)$ be the expected cost of destroying N_p primary targets under policy D, given also that there are N_s secondary targets. As usual let D* be the optimal policy, and $\mathcal{D}$ be the family of all allocation policies. We denote:

$$C(N_s,N_p) = C_{D*}(N_s,N_p) = \min_{D\in\mathcal{D}} C_D(N_s,N_p).$$

Now we make the following observation: as long as $N_s \leq N_s^*(N_p)$, we have $D^*(N_s,N_p) = AP$. Only anti-primary missiles are in use, and the probability of hitting a target is $q(N_s)\cdot P_p$. The expected number of anti-primary missiles, to hit a single primary target, is obviously $1/P_p\cdot q(N_s)$, and so the expected cost of destroying N_p primary targets is

$$C(N_s,N_p) = \frac{C_p\cdot N_p}{P_p\cdot q(N_s)} \quad \text{for} \quad N_s \leq N_s^*(N_p).$$

Now consider the state $(N_s^*(N_p)+1,N_p)$. It is implied by the definition of $N_s^*(N_p)$ that the optimal decision at that state is anti-secondary. As a result of such a decision, two transitions are possible, as shown here:

$$(N_s^*(N_p)+1,N_p) \begin{cases} \nearrow (N_s^*(N_p),N_p) & \text{Next optimal decision is anti-primary} \\ \searrow (N_s^*(N_p)+1,N_p) & \text{Next optimal decision is again anti-secondary.} \end{cases}$$

The probability of making the upper transition in a single step is $P_s\cdot q(N_s^*(N_p)+1)$, so that it will take--in expectation-- $1/P_s\cdot q(N_s^*(N_p)+1)$ missiles to make the upper transition. Once

this transition has been made, it will be optimal to launch only anti-primary missiles until the mission is completed. Therefore, we find that

$$C(N_s^*(N_p)+1, N_p) = \frac{C_s}{P_s \cdot q(N_s^*(N_p)+1)} + \frac{C_p \cdot N_p}{P_p \cdot q(N_s^*(N_p))} .$$

Since this is the minimal expected cost of destroying all the primary targets, when starting from state $(N_s^*(N_p)+1, N_p)$, this should be less than the cost that would have resulted had the attacker decided to use only anti-primary missiles from the beginning. So, we write:

$$\frac{C_s}{P_s \cdot q(N_s^*(N_p)+1)} + \frac{C_p \cdot N_p}{P_p \cdot q(N_s^*(N_p))} < \frac{C_p \cdot N_p}{P_p \cdot q(N_s^*(N_p)+1)} . \quad \text{(III.33)}$$

The fundamental observation is that $N_s^*(N_p)$ should be, by definition, the minimal value of N_s to satisfy Inequality (III.33). This is the clue to the solution; $N_s^*(N_p)$ is given by:

$$N_s^*(N_p) = \min\{N_s : \frac{C_s}{P_s \cdot q(N_s+1)} + \frac{C_p \cdot N_p}{P_p \cdot q(N_s)} < \frac{C_p \cdot N_p}{P_p \cdot q(N_s+1)}\} . \quad \text{(III.34)}$$

Note that $N_s^*(N_p)$ may very well be equal to zero, which is to say that even with the presence of <u>one</u> secondary target, it is preferable to destory the secondary target first.

The function $N_s^{**}(N_p)$ is now derived by an argument very similar to the one used in deriving Eq. (III.33). First notice

that we can explicitly write $C(N_s,N_p)$ for values N_s such that $N_s^*(N_p) < N_s \leq N_s^{**}(N_p)$.

$$C(N_s,N_p) = \sum_{n=N_s^*(N_p)+1}^{N_s} \frac{C_s}{q(n)\cdot P_s} + \frac{N_p\cdot C_p}{q(N_s^*(N_p))\cdot P_p} \qquad \text{(III.35)}$$

This follows from the fact that the attacker has to hit $N_s-N_s^*(N_p)$ secondary targets before he switches to an anti-primary course. The first term of Eq. (III.35) represents the expected cost of doing this. The second term is then the expected cost of killing the N_p primary targets. Notice that $q(N_s^*(N_p))$ appears in the denominator of the second term, since at the switching stage, exactly $N_p^*(N_s)$ secondary targets will still be alive.

Since Eq. (III.35) expresses the optimal expected cost only for values of N_s such that $N_s \leq N_s^{**}(N_p)$, it is obvious that that expression should yield values smaller than the expected cost associated with any other policy. Specifically, we examine the "Anti-primary only" policy. Its expected cost is $C_p\cdot N_p/q(N_s)\cdot P_p$, so that for all $N_s \leq N_s^{**}(N_p)$ we have:

$$C(N_s,N_p) < \frac{C_p\cdot N_p}{q(N_s)\cdot P_p} \qquad \text{(III.36)}$$

In fact, the least N_s to violate Eq. (III.36) is $N_s^{**}(N_p)+1$ (see definition of $N_s^{**}(N_p)$). We therefore conclude that:

$$N_s^{**}(N_p) = \mathrm{Min}\{N_s: \; N_s > N_s^*(N_p) \text{ and } \frac{C_p\cdot N_p}{q(N_s+1)\cdot P_p} \leq \sum_{n=N_s^*(N_p)+1}^{N_s+1} \frac{C_s}{q(n)\cdot P_s} + \frac{N_p\cdot C_p}{q(N_s^*(N_p))\cdot P_p}\} \qquad \text{(III.37)}$$

It is possible that $N_s^{**}(N_p)$ will be ∞. This will mean that for all values N_s such that $N_s > N_s^*(N_p)$, we have to hit $N_s - N_s^*(N_p)$ secondary targets first, and then switch to the primary target.

2. <u>Examples</u>

We shall present three examples corresponding to three different survival functions:

a) $q(N_s) = q_0^{N_s}$ (Independent operations).

b) $q(N_s) = \begin{cases} 1 & \text{if } N_s = 0 \\ q_0 = \text{const.} & \text{if } N_s > 0 \end{cases}$

c) $q(N_s) = q_0 + (1-q_0)\cdot(1-r)^{N_s}$

The operational interpretation of these three functions has been explained in Section III.B.

a) $q(N_s) = q_0^{N_s}$

We substitute in Eq. (III.33) and get:

$$\frac{C_s}{P_s \cdot q_0^{N_s+1}} + \frac{C_p \cdot N_p}{P_p \cdot q_0^{N_s}} < \frac{C_p \cdot N_p}{P_p \cdot q_0^{N_s+1}}$$

or, equivalently:

$$\frac{C_p \cdot N_p \cdot q_0}{P_p} < \frac{C_p \cdot N_p}{P_p} - \frac{C_s}{P_s}$$

We see that the validity of the inequality doesn't depend upon N_s at all. Therefore $N_s^*(N_p) = 0$ or $N_s^*(N_p) = \infty$, depending upon

the value of the parameter q_0. We easily deduce:

$$\text{(A)} \qquad q_0 < 1 - \frac{C_s \cdot P_p}{C_p \cdot P_s \cdot N_p} \Longrightarrow N_s^*(N_p) = 0$$

$$\text{(B)} \qquad q_0 \geq 1 - \frac{C_s \cdot P_p}{C_p \cdot P_s \cdot N_p} \Longrightarrow N_s^*(N_p) = \infty \,.$$

Since q_0 is a number which has an interpretation of probability, we have $0 \leq q_0 \leq 1$, and so, if

$$N_p \leq \frac{C_s/P_s}{C_p/P_p} \qquad \text{(III.38)}$$

then $1 - (C_s \cdot P_p)/(C_p \cdot P_s \cdot N_p) \leq 0$, and condition (B) above is always the valid one, so that $N_s^*(N_p) = \infty$. This result has a very clear operational interpretation: If there are only a few primary targets, it is always preferable to deal only with them, and ignore the secondary targets. The question is of course how to interpret quantitatively the term "few". Eq. (III.38) gives the answer: The number of primary targets should be less than the ratio of expected cost to destroy one secondary target in ideal conditions (C_s/P_s) to the expected cost to destroy one primary target in ideal conditions (C_p/P_p). This ratio can be viewed as the 'worth' of one secondary target measured in 'units' of the "worth" of a primary target. If the number of primary targets is less than the "worth" of one secondary target, then it is not profitable to deal with secondary targets at all.

If, however, $N_p > C_s \cdot P_p / C_p \cdot P_s$, then we may define:

$$s = 1 - \frac{C_s \cdot P_p}{C_p \cdot P_s \cdot N_p} . \qquad \text{(III.39)}$$

If $q_0 < s$, then $N_s^*(N_p) = 0$ so that if there is one secondary target, it should be attacked first. If $q_0 \geq s$, only the primary targets should be attacked. Notice that we can't say anything yet about the case $N_s = 2$, since we have to calculate $N_s^{**}(2)$ first. We find $N_s^{**}(N_p)$ using (III.37):

$$\frac{C_p}{q_0^{N_s+1} \cdot P_p} \leq \sum_{n=1}^{N_s+1} \frac{C_s}{q_0^n \cdot P_s} + \frac{N_p \cdot C_p}{P_p} = \frac{C_s}{P_s} \cdot \frac{1-q_0^{N_s+1}}{(1-q_0) \cdot q_0^{N_s+1}} + \frac{N_p \cdot C_p}{P_p} .$$

Using simple algebraic operations this inequality can be shown to be equivalent to:

$$1 - \frac{C_s \cdot P_p}{C_p \cdot P_s \cdot N_p} = s \leq q_0$$

In this form, the inequality does not depend on N_s! Therefore, if the parameters of the problem satisfy this inequality (case B above), then from definition (III.37), and using the fact we have just proven that $N_s^*(N_p) = \infty$ in this case, we conclude that $N_s^{**}(N_p) = \infty$ also (and it does not really have any significance here!). If, on the other hand, the parameters of the problem do not satisfy this equation, in which case we have seen that $N_s^*(N_p) = 0$, then no value of N_s satisfires the above inequality (or, as can be stated formally, using

definition (III.37), $N_s^{**}(N_p) = \infty$). This means that no matter how many secondary targets there are, the attacker should attack them first until they are all destroyed.

(b) $q(N_s) = q_0 = \text{const for } N_s > 0,\ q(0) = 1$

This case, which is very hard to handle analytically using other criteria (MPH, MEMP criteria) is of no great difficulty if the MEC criterion is used. It is an interesting case for which a simple argument leads to the conclusion that $N_s^*(N_p)$ must be either 0 or ∞. The argument is that if the attacker decides to attack secondary targets, he must be determined to attack all of them before switching to the primary target, because no reduction in attrition will be gained, unless he so behaves. Mathematically speaking, the following statement must be true: It is impossible that for some N_p we'll have two values, N_s^1, N_s^2, such that $0 < N_s^1 < N_s^2$, and

$$D^*(N_s^2, N_p) = AS$$

$$D^*(N_s^1, N_p) = AP$$

This simply means that $N_s^*(N_p)$ is either 0 or ∞. We now calculate the condition for each case. We just carry out the formal operation, given in (III.34), and discover immediately that these conditions are identical with those found in the previous example. That is:

If $N_p \le \dfrac{C_s \cdot P_p}{C_p \cdot P_s}$ $\quad N_s^*(N_p) = \infty \Rightarrow$ always shoot anti-primary missiles

If $N_p > \frac{C_s \cdot P_p}{C_p \cdot P_s}$, then: If $q_0 \leq s$, then $N_s^*(N_p) = 0$

If $q_0 > s$, then $N_s^*(N_p) = \infty$

where s is the parameter defined before (Eq. (III.39)).

The difference between this example and the previous one is in the value of $N_s^{**}(N_p)$. In example a), $N_s^{**}(N_p) = \infty$ always. To calculate it for this case we proceed formally, using (III.37). We have to find the least N_s which satisfies the following inequality:

$$\frac{C_p \cdot N_p}{q_0 \cdot P_p} < \frac{C_s \cdot N_s}{q_0 \cdot P_s} + \frac{N_p \cdot C_p}{P_p} \tag{III.40}$$

Notice that we substitute in (III.37) the values $N_s^*(N_p) = 0$, because only in this case is there a meaning for $N_s^{**}(N_p)$. Therefore, it is assumed here that $q_0 \leq s$.

The first N_s to satisfy inequality (III.40) is:

$$N_s^{**}(N_p) = \left[\frac{(1-q_0) \cdot N_p}{\frac{C_s \cdot P_p}{P_s \cdot C_p}} \right] + 1 \,, \tag{III.41}$$

$$= \left[\frac{1-q_0}{1-s} \right] + 1$$

where s is defined in Eq. (III.39). Since we assume $q_0 \leq s$, we deduce from Eq. (III.41) that if $N_s^*(N_p) = 0$, then:

$$N_s^{**}(N_p) \geq 2$$

This last result is a very interesting one. It says that $N_s^*(N_p) = 0$ implies $N_s^{**}(N_p) \geq 2$. In other words, if the

parameters of the problem are such that it is optimal to deal with the secondary target, when there is only one secondary target, then it must be optimal to deal with secondary targets first when there are two secondary targets also. This fact is somewhat strange, since it is hard to base it on some common sense or intuitive argument. (It recalls a similarly strange fact we found in Chapter II, namely that $M^*(1) \geq 2$ always for the MPH allocation problem with one secondary target).

(c) $q(N_s) = q_0 + (1-q_0)r_0^{N_s}$

This example is presented to prevent the impression which might have been created by the last two examples, that $N_s^*(N_p)$ is "usually" zero or ∞. In fact, this survival function can be considered a generalization of the functions treated in (a) and in (b): By putting $q_0 = 0$ we get case (a), and by putting $r_0 = 0$ (with the convention $0^0 = 1$) we get case (b). By substituting this survival function into Eq. (III.33) we have:

$$\frac{C_s}{P_s\cdot[q_0+(1-q_0)\cdot r_0^{N_s+1}]} + \frac{C_p\cdot N_p}{P_p\cdot[q_0+(1-q_0)\cdot r_0^{N_s}}$$

$$< \frac{C_p . N_p}{P_p\cdot[q_0+(1-q_0)\cdot r_0^{N_s+1}]}$$

If we put $s = 1 - (C_s\cdot P_p)/C_p\cdot P_s\cdot N_p$ as before, it can be shown that the above inequality is equivalent to:

$$r_0^{\,N} < \frac{(1-s)\cdot q_0}{(1-q_0)\cdot(s-r_0)} \tag{III.42}$$

so that $N_s^*(N_p) = \mathrm{Min}\{N_s\colon$ inequality (III.42) is satisfied$\}$.

There are three different cases now:

(1) $s < r_0$, in which case the right-hand side of Eq. (III.42) is negative, and it has no finite solution N_s, which is to say that $N_p^*(N_s) = \infty$ (always shoot at the primary!)

(2) If $\frac{(1-s)\cdot q_0}{(1-q_0)\cdot(s-r_0)} > 1$, then $N_s^*(N_p) = 0$ (which means that at least for $N_s = 1$, the attacker should destroy the secondary target first.

(3) If $0 < \frac{(1-s)\cdot q_0}{(1-q_0)\cdot(s-r_0)} \leq 1$, then:

$$N_s^*(N_p) = \left[\frac{\ln \frac{(1-s)\cdot q_0}{(1-q_0)\cdot(s-r_0)}}{\ln r_0}\right] + 1 .$$

Here $N_s^*(N_p)$ is a finite positive number: Let us look at a numerical example: suppose $C_p = C_s$, and $P_p = P_s$, and let $N_p = 5$, q_0 (which has the meaning of survival probability given that one defense unit is shooting at the attacking missile) = 0.5, r_0 ($1-r_0$ = reliability of each defense unit) = 0.25. We have, therefore, $s = 0.8$, and so we fall into category (c) and

$$N_s^*(5) = \left[\frac{\ln \frac{0.2\times 0.5}{0.5\times(0.8-0.25)}}{\ln 0.25}\right] + 1$$

$$= [0.73] + 1 = 1$$

That is, if there are 5 primary targets, then as long as there are no more than one secondary target to defend them, the attacker should ignore the secondary. If $N_s = 2$ he should destroy one secondary target first. If $N_s = 3$, nothing can be concluded yet. We must calculate $N_s^{**}(5)$ first.

IV. OPTIMAL MISSILE DEPLOYMENT GAME MODELS INVOLVING THE USE OF A 'CAUTIOUS' MODE OF DEFENSIVE OPERATION

A. INTRODUCTION

In developing the models of Chapters II and III, we have assumed that in countering an offensive missile the defense always responds in the same way. That is, it always attempts to intercept an arriving offensive missile by launching a defensive missile.

In reality the secondary targets can quite simply immunize themselves against being hit by anti-secondary missiles, by operating the units in a special deceptive mode. For example, if the anti-secondary missile is using an anti-radiation (AR) guidance head, the defense unit can temporarily shut off the radar electromagnetic radiation, thus denying the missile the signal necessary for its proper guidance. If the missile guidance relies on some sort of electro-optic or infra-red imaging, then deploying dummy targets, concealment devices and other deceptive methods may very well provide the secondary target with an almost perfect protection against the missile.

In this chapter we therefore assume that upon each arrival of a new offensive missile, the defense may decide to work in one of two operational modes:

(a) Mode 1--which is the "normal" mode,

(b) Mode 2--which is a special mode of caution against anti-secondary targets. Usually some deception method is involved in this mode.

It would be quite accurate to state that in Mode 2 the defense target, while being much less vulnerable to an arriving missile, is also much less effective in intercepting this missile. For instance, in the example of AR missiles mentioned above, switching off the radar renders the secondary target itself almost incapable of detecting the missile and intercepting it. When the defender detects a missile, he thus faces the dilemma of which mode to select for the system operation. Had he known that the missile is an anti-secondary one (i.e., aimed at the defense unit) he would obviously have selected Mode 2. But he usually cannot be certain about the missile destination. Therefore, so long as the defender cannot eliminate the uncertainty about the missile destination, he cannot avoid the possibility of making the "wrong" decisions. (By "wrong" we mean here, either operating on Mode 1 when the missile is anti-secondary, or operating on Mode 2 when the missile is actually anti-primary). Assuming that the defender pursues an optimal course of behavior in that respect, the attacker also has a dilemma of which type of missile to launch at every stage of the process. If we adopt the assumption that both the defender and the offender are aware of the choices available to each other, and further, that they are strictly opposed in their objectives, then we are very naturally led to formulating the situation as a zero-sum game. Since we are dealing with dynamic processes which progress in stages, we don't actually have a simple unique two-by-two matrix game,

but rather a sequence of games, related to each other. Such sequences of games are called in the literature stochastic games.

Section B of this chapter gives a very brief review of the theory and applications of stochastic games, emphasizing mainly the relation to dynamic programming. In Section C we list and explain in detail the assumptions which underlie the models. We also introduce the notation. Section D examines and criticizes the applicability and relevance of zero-sum stochastic models to real tactical decision-making processes. Sections E-G deal with detailed mathematical solutions of the games which arise in our case, each section devoted to one of the three criteria we use in this thesis (MPH, MENP, MEC criteria). We give general theorems about the structure and properties of optimal policies of both defender and attacker, and then show the solutions to each case.

B. A BRIEF REVIEW OF STOCHASTIC GAMES THEORY AND APPLICATIONS

The two person zero-sum stochastic game was first introduced in an elegant paper by Shapley [5], where he presents the basic notion of a stochastic game and the theory which underlies the computational methods for calculating the value-vector and the set of optimal strategies for each player.

Following Shapley, a stochastic game is defined as a sequential process in which the players step from position to position according to transition probabilities controlled jointly by the two players. It is assumed that there are a

finite number, M, of positions, and finite sets of actions available to each player at each position. Let A_1, A_2 be the set of actions available to player 1 and player 2, respectively. If, at position (state) k, player 1 chooses action i and player 2 chooses action j, then there is an immediate payment $a^k_{i,j}$ from player 2 to player 1, and in addition, there is a probability $P^{k\ell}_{ij}$ that the game moves to the new position, ℓ. The process therefore is determined by the initial state and by the following M^2+M matrices:

$$P^{k\ell} = (P^{k\ell}_{ij} | i = 1,2,\ldots,|A_1|, j = 1,2,\ldots,|A_2|)$$

for $k,\ell = 1,2,\ldots,M$, and

$$A^k = (a^k_{ij} | i = 1,2,\ldots,|A_1|, j = 1,2,\ldots,|A_2|)$$

for $k = 1,2,\ldots,M$, where obviously:

$$0 \leq P^{k\ell}_{ij} \leq 1$$

and

$$\sum_{\ell=1}^{M} P^{k\ell}_{ij} \leq 1$$

The number $1 - \sum_{\ell=1}^{M} P^{k\ell}_{ij} = s^k_{i,j}$ is the probability that the game will terminate after one step, given that it is in state k and players 1 and 2 choose actions i and j respectively.

Shapley concentrated on games which terminate, with probability one, after a finite number of stages. Obviously, a

sufficient condition for that is

$$s = \min_{i,j,k} s_{i,j}^{k} > 0.$$

The stochastic games which we shall treat in this thesis do not satisfy this condition. They do however terminate with probability one.

State k is denoted by a symbol Γ^k which is to be regarded as "the sequential game the players start to play when they are in position k". The stochastic game is sometimes considered as a collection of game elements $\{\Gamma^k: k = 1,2,...,M\}$. To each game element there corresponds a matrix of payoffs A_k, in which the (i,j) entry has the form

$$a_{i,j}^{k} + \sum_{\ell=1}^{M} P_{ij}^{k\ell} \cdot \Gamma^{\ell}$$

The entries of the matrics A_k $(k = 1,2,...,n)$ are therefore mixtures of real rewards and game elements. Let $\vec{\alpha}$ be an M-dimensional vector of numbers $\alpha_1,\alpha_2,...,\alpha_M$. Shapley denotes by $A_k(\vec{\alpha})$ the numerical matrix obtained when the game elements $\{\Gamma^\ell: \ell = 1,2,...,M\}$ appearing in the entries of the matrix A_k are replaced by the numerical values $\alpha_1,\alpha_2,...,\alpha_M$. Then he defines the following transformation T which maps the M-dimensional Euclidean space into itself:

$$T\vec{\alpha} = (\text{val } A_1(\vec{\alpha}), \text{val } A_2(\vec{\alpha}), ..., \text{val } A_M(\vec{\alpha})). \qquad \text{(IV.1)}$$

The vector $T\vec{\alpha}$ is the vector of the minimax values of the matrices $\{A_T(\vec{\alpha}): \ k = 1,2,\ldots,M\}$.

Shapley shows that the condition of certain termination mentioned above guarantees that the stochastic game has a unique value-vector, which is the solution of the following set of equations:

$$T\vec{\phi} = \vec{\phi} \qquad \text{(IV.2)}$$

in the vector of unknowns $\vec{\phi}$. In fact, the role that the certain termination condition plays here is just to make T a contraction transformation so that Banach Theorem on contraction mapping applies and determines the uniqueness of the solution to the system (IV.2).

If $\vec{\phi}^o$ is the solution to Eq. (IV.1) then ϕ_k^o is the value (in the usual Von-Neumann-Morgenstern sense) of the game element Γ^k. Shapley further shows that sets of optimal strategies of both players, in state k, are the same sets of optimal strategies associated with the game for which the payoff matrix is $A_k(\vec{\phi}^o)$ ([5], p. 1097, Theorem 2).

Everett [8] and Gillette [9] in 1957 generalized the results of Shapley by relaxing the condition that termination occurs with probability one. In the general case there is a positive probability that the process will run indefinitely, thus a revision of the definition of the value is necessary (since a simple summation of rewards on all individual stages may give infinite results). In Gillette [9], two different approaches to the problem of value definition are considered,

namely, discounting and averaging. These two methods are most commonly used in the literature of infinite dynamic programming processes and infinite games. They are used to avoid having infinite values (payoffs) as a result of summation over all immediate rewards. Gillette's main concern is the existence of stationary optimal strategies for both players, under each of the above modified payoff functions (which he calls "effective" payoffs). Everett's paper deals with various "existence" questions and mathematical properties of solutions to the most generalized recursive game.

Hoffman and Karp [10] considered the case of a nonterminating game, i.e., a stochastic game in which

$$\sum_{\ell=1}^{M} P_{ij}^{k\ell} = 1 \quad \text{for all} \quad i,k,j.$$

Some papers ([11],[12] are the prominent ones) were written about techniques and algorithms to solve the asymptotic value equations (see (IV.2) above), of infinite games. Most of them elaborate on variations of the basic successive approximations technqiue (which was suggested by Shapley himself in [5]).

All the stochastic games which we actually solve in this thesis are known to terminate in a finite number of stages. In two of our criteria, namely, the MPH and MENP criteria, the maximum number of stages the process is allowed to continue is prescribed (that is, we solve <u>truncated</u> stochastic games). In such processes, the maximum number of stages which are left in a given state is one of the parameters which

determine the state. Therefore there is no meaning to stationarity in these cases, and clearly the optimal strategies will depend on time.

It is interesting to note that the whole subject of Markovian decision processes (or stochastic Dynamic Programming) can be viewed simply as a special case of the stochastic games theory: If we consider one player as "dummy", that is, a player who has only one pure action available, then the problem for the other player will be a Dynamic Programming problem. To see this, let us consider again Equation (IV.1)

$$T\vec{\alpha} = (\text{val } A_1(\vec{\alpha}), \text{val } A_2(\vec{\alpha}), \ldots, \text{val } A_M(\vec{\alpha})).$$

If the matrices $A_k(\vec{\alpha})$ $(k = 1,2,\ldots,M)$ above are $1 \times |A_2|$, as is the case when player 1 is "dummy" and player 2 has $|A_2|$ pure actions available, then we can write

$$\text{val } A_k(\vec{\alpha}) = \min_{j \in A_2} [a_j^k + \sum_{\ell=1}^{M} P_j^{k\ell} \alpha_\ell] \tag{IV.3}$$

where we've written $a_j^k (P_j^{k\ell})$ instead of $a_{1,j}^k (P_{1,j}^{k,\ell})$, since player 1 has just one possible action. Now we combine Eq. (IV.3) with Eq. (IV.2) and find, that in the infinite Dynamic Programming process, the equation which we have to solve is the following:

$$\phi_k = \min_{j \in A_2} [a_j^k + \sum_{\ell \in A_2} P_j^{k\ell} \phi_\ell] \tag{IV.4}$$

which is exactly the well-known fundamental equations of the Markovian decisions processes (see Dreyfus & Law [14, page 175]. The ϕ_k is then the optimal payoff of the process given that it starts from state k. If we deal with stochastic dynamic processes in finite time, then α_ℓ of Eq. (IV.3) above should stand for the optimal payoff for a process of n stages (for some n) while val $A_k(\vec{\alpha})$ is the optimal payoff for a process of n+1 stages. If we denote by ϕ_k^n the optimal payoff of a Markovian decision process which starts at state k, and lasts n units of time (or, <u>at most</u> n units of time), then Eq. (IV.3) should be interpreted as:

$$\phi_k^{n+1} = \min_{j \in A_2} [a_j^k + \sum_{\ell=1}^{M} P_j^{k\ell} \cdot \phi_\ell^n]$$

which, apart from differences in notation, is the same as the general recursive equation of Markovian decision processes, as given, for instance, in Dreyfus & Law [14, p. 174, Eq. 13.4].

As to applications of stochastic games theory to real world processes, it seems astonishing that almost no significant use has been made of it. The books by Luce and Raiffa [13] and by Owen [4] mention some interesting problems for which the stochastic game is a very natural formulation. The main ones are:

(a) Games of Survival--which are applicable mainly to gambling theory (see Luce & Raiffa [13, Appendix A.4].

(b) Exhaustion Games--which can serve to model inspection processes, evasion and search processes (see example

in Owen [4, Ch. V.2]. In the second part of this thesis we introduce a very plausible application of exhaustion games in the evaluation of the effectiveness of decoys in missile warfare.

(c) Allocation of Military Forces--these problems are usually titled under the term "Blotto Games", and were treated quite thoroughly in the literature (see Dresher [15], and Blackett [16]). It should be said, however, that Blotto games are not necessarily sequential games.

Very few articles were written on applications of stochastic games theory to the analysis of duels. Charnes and Schroeder [7] claim to analyze tactical antisubmarine duels through the theory of stochastic games. Their article, however, contains very little about the antisubmarine warfare application. It mainly repeats the general theory and analyzes some computational procedures of the general recursive equations of the theory.

Sweat ([17],[18]) presents very attractive models for a duel between an attacker and a defender. The models of Sweat were probably motivated by undersea warfare problems, but they contain many elements which make them equally applicable to other types of warfare. We make use of some of Sweat's ideas in this chapter as well as in Part II of the thesis, where we analyze problems involving decoys. In the following, we discuss briefly the work of Sweat, and its relation to this thesis.

In [17] Sweat deals with the following problem: A duel is initiated by an attacker at time $t = -T$. At some instant in the interval $[-T,0]$, the attacker has to launch his (single) weapon. He always knows the current number of weapons the

defender has. If the defender has k weapons, then his probability of survival is P_k if he responds with one of his k weapons right at the time the attacker launches his weapon, and is q_k ($<P_k$) if he responds after the weapon has been launched (we use here the notation of Sweat). The defender may or may not detect the attacker at the time of attack, and may also detect false targets. The process of false targets detection is assumed to be a non-homogeneous Poisson process on the interval $[-T,0]$, with rate function $\lambda(t)$, known to the attacker. The defender classifies a detected object as a 'real' or 'false' target. Therefore two types of errors are possible and the probability of each is known to both players. The payoff is the defender's probability of survival at time $t = 0$.

From the description of the problem it is clear that the defender desires to form a policy of response to a detection and classification of a target in a manner that will provide high probability of detecting correctly the true target (i.e., the attacker's weapon) when it is launched, and at the same time will keep the chance of being exhausted of weapons prior to the attack as low as possible. The desire of the attacker is, of course, just the opposite one. He would try to exploit the presence of false targets in order to reduce as much as possible the chances that the defneder would detect his weapon correctly when he still has weapons in his stockpile. The way to achieve this is to "wait" and let the defender consume

his weapons on false targets. He cannot, however, wait too long if the defender has relatively many missiles, because as the process runs short of time, the tendency of the defender will be to shoot at all targets (false and true). The problem of choosing the optimal time to launch for the attacker is thus quite complicated.

Sweat solves this problem using a stochastic game in a continuous time. There is no conceptual difference between a stochastic game in continuous time and the games in discrete time that were reviewed before. In fact, the game with continuous time is approximated as a game in discrete time by dividing the time interval $[-T,0]$ into small segments of length Δt. Writing then recursive equations for the value of the game in the approximating problem, and letting $\Delta t \to 0$, Sweat thus derives an iterative system of first-order differential equations. The unique solution of that system is a vector $(V_1(t),\ldots,V_k(t))$, where $V_i(t)$ $(i = 1,2,\ldots,k)$ is the value of the game which starts at time t, with the defender having i weapons left.

In his second paper, Sweat [18] modifies this problem by allowing the attacker to be one of two different types, with different probabilities of detection and classification, and with different probabilities of killing the defender. The same concepts and methodology are used in solving this problem as were used in [17].

C. THE ANTI-SECONDARY/ANTI-PRIMARY ALLOCATION (ASAPA) GAME MODELS--GENERAL DESCRIPTION

In this section we present in detail the assumptions and concepts of our game models which we consider quite adequate in describing a situation in which secondary targets have some simple way of protecting themselves from anti-secondary missiles (other than shooting at them). This model has the following advantages over the models that were discussed in Chapters II and III:

(1) It refers to the defender as a conscious player,

(2) It assumes that the defender is capable, to some extent, of distinguishing between anti-primary and anti-secondary missiles.

The way in which distinguishability is reflected in our models, is to assume existence of the classification probabilities (as Sweat [17] does. See also Gorfinkel [19]). We introduce the following notation:

α_p = Probability that a detected anti-primary missile will be classified as anti-primary.

α_s = Probability that a detected anti-secondary missile will be classified as anti-secondary.

Obviously, $1-\alpha_p$ ($1-\alpha_s$) is the probability that a detected anti-primary (anti-secondary) missile will be classified as an anti-secondary (anti-primary) missile. We assume that the probability of detection is 1.

We assume that at each stage the attacker can choose to launch either an anti-primary or anti-secondary missile. The defender, after classifying the missile, selects his mode of

operation between Mode 1 ("normal" mode) and Mode 2 ("cautious" mode). We assume that:

(a) The probabilities of survival of the attacking missile when the defender is using mode 1 and mode 2, are q_1 and q_2, respectively.

(b) In mode 2, the secondary target is absolutely invulnerable to the anti-secondary missiles. (That is, the probability of hit, even if the missile survives, is zero.)

(c) In mode 2 the secondary target is less effective than in mode 1. The mathematical expression of that assumption is clearly

$$\boxed{q_1 < q_2}$$

We assume that the defender selects, at each stage, a response program, that is, a program which dictates in which mode he will operate his system at each of the two possible classification outcomes.

Four response programs are possible corresponding to the four different combinations of offensive missile types and operation mode of the defense. We shall use the following notation for them:

P_1-S_1 = Use mode 1 regardless of the classification; that is, use mode 1 regardless of whether the offensive missile is classified as anti-primary or as anti-secondary.

P_1-S_2 = Use mode 1 if the missile is classified as anti-primary, and mode 2 if the missile is classified as anti-secondary.

P_2-S_1 = Use mode 2 if the missile is classified as anti-primary, and mode 1 if it is classified as anti-secondary.

P_2-S_2 = Use mode 2 in either case.

It will always be assumed that both the attacker and the defender have perfect information about the state of the process. As in Chapters II and III we use the notation P_p (P_s) for the single-shot-probability of kill by an anti-primary (anti-secondary) missile, given that the missile survived the interception attempt of the defense.

We shall restrict ourselves to the case of one secondary target only. The case of several secondary targets requires very heavy and awkward technical, mathematical work, but no new conceptual difficulties are encountered. Therefore we concentrate on the investigation of the case where only one secondary target is present.

Notice that no consideration is made about the size of the defender's stockpile. We assume that no constraint does really exist in this aspect, that is, the defender always has enough defensive weapons so that he can counter the offensive missile. This assumption is quite realistic, especially when reloading times are far shorter than the characteristic time elapsing between launches of the offensive missiles. Even if the defender did have a limit on the number of missiles he has, it would not raise any significant difficulties to our models since we can define the number of missiles he has as the state variable (dictating the maximum number of stages of the game) and nothing in our models would be changed (given only that we used either the MPH or MENP criterion). In Part II of the thesis, where decoys are considered, we shall formulate and

solve problems in which the limit on number of defensive missiles does play a role in the model.

We are now ready to formulate the games and solve them. Before doing that, some comments are appropriate on the adequacy of the game methodology to our problem.

D. COMMENTS ON THE ADEQUACY OF GAME MODELS TO MISSILE ALLOCATION PROBLEMS

A zero-sum game is, by its very nature, a mathematical idealization of a conflict taking place between two players of strictly contradictory interests. The degree to which results based on such models are relevant, either to the description of real processes involving human decisions or to prescription of optimal course of behavior for any of the players, is very controversial. The philosophy of game theory is very hard to accept in the real decision-making world, since in the basics of this philosophy lies the assumption that both players are extremely sophisticated and that both are aware of game theory itself and believe that so is their rival. The fact that optimal behavior, as dictated by game models, very often prescribes randomized strategies, makes it even harder, technically and mentally, to implement in the real world. For example: suppose that the optimal defensive policy in our problem, in some specific case, is to randomize over P_1-S_1 and P_1-S_2 response programs. It is very hard to imagine how does a human, who is responsible for the defense, actually follow this prescription. The randomization action as a

procedure to be performed in real processes is something which is very much rejected by natural instincts. Hence, it should be clear that the models given hereby should not be interpreted as either descriptive or prescriptive models. As Luce & Raiffa [13, page 63] say:

> ...It is crucial that the social scientist recognize that game theory is not descriptive, but rather (conditionally) normative. It states neither how people do behave nor how they should behave in an absolute sense, but how they should behave if they wish to achieve certain ends. It prescribes for given assumptions courses of action for the attainment of outcomes naving certain formal "optimum" properties...

In addition to the importance of the game model, as pointed out by Luce & Raiffa, we would emphasize yet another gain from these models. This is the fact that they provide estimates for the "min max" values of the relevant measure of effectiveness (or cost). The value of a zero sum game represents, as is well known, the "best worst" case for both players. It tells each player what is the minimum benefit (or maximum loss) which theoretically can be guaranteed. Having the value is therefore amounts to having a very significant, although partial only, information to both players.

E. THE ANTI-SECONDARY/ANTI-PRIMARY ALLOCATION (ASAPA) GAME MODEL WITH MAXIMUM EXPECTED NUMBER OF PENETRATORS (MENP) PAYOFF

We denote by Γ_1^M the game played when the secondary target is alive, and there are M missiles yet to be launched by the attacker. The symbol Γ_0^M stands for the game starting with the secondary target already destroyed. We assume that the

payoff is the maximum expected number of penetrators (MENP criterion). Clearly, we have:

$$\mathrm{val}(\Gamma_0^M) = M$$

The game Γ_1^M can be described by a matrix the entries of which are combinations of real reward and some distribution on the game elements Γ_0^{M-1} and Γ_1^{M-1} (see Section B). The matrix of the game element Γ_1^M is the following:

Defender's choices \ Attacker's choices	Attacker uses anti-primary missile (AP)	Attacker uses anti-secondary missile (AS)
P_1-S_1 (*)	$q_1 + \Gamma_1^{M-1}$	$P_s \cdot q_1 \cdot \Gamma_0^{M-1} + (1-P_s \cdot q_1)\Gamma_1^{M-1}$
P_1-S_2 (*)	$\alpha_p \cdot q_1 + (1-\alpha_p) \cdot q_2 + \Gamma_1^{M-1}$	$(1-\alpha_s) \cdot q_1 \cdot P_s \cdot \Gamma_0^{M-1} + [1-(1-\alpha_s) \cdot q_1 \cdot P_s]\Gamma_1^{M-1}$
P_2-S_1 (*)	$\alpha_p \cdot q_2 + (1-\alpha_p) \cdot q_1 + \Gamma_1^{M-1}$	$\alpha_s \cdot q_1 \cdot P_s \cdot \Gamma_0^{M-1} + [1-\alpha_s \cdot q_1 \cdot P_s]\Gamma_1^{M-1}$
P_2-S_2 (*)	$q_2 + \Gamma_1^{M-1}$	Γ_1^{M-1}

(*) See Section C for definitions.

To make things clear we explain in details two of the entries of the above matrix.

Consider first the entry P_1-S_2--AP above. It corresponds to the attacker launching an anti-primary missile. There is a probability α_p that it will be classified as anti-primary. Hence Mode 1 will be selected by the defender (as the P_1-S_2 response program dictates), and the probability of penetration of the launched missile will then be q_1. If the missile is classified as anti-secondary, mode 2 will be selected, and the probability of penetration is then q_2. The unconditional probability of survival of the missile is thus $\alpha_p q_1 + (1-\alpha_p) q_2$. In addition, the next game to be played is Γ_1^{M-1} (with probability one).

Now we explain entry P_1-S_2--AS. Here the attacker launches an anti-secondary missile. If it is classified correctly, then there is zero probability of killing the secondary target (by the assumption we made on mode 2). If it is classified incorrectly (that is, as anti-primary), as it will be with probability $1-\alpha_s$, then it has probability $q_1 \cdot P_s$ of killing the secondary target. The unconditional probability of killing the secondary target is thus $(1-\alpha_s)q_1 \cdot P_s$, and then the next game is Γ_0^{M-1}. The probability of not killing the target is $1-(1-\alpha_s)q_1 P_s$, and given that, the next game is Γ_1^{M-1}. Therefore, the entry is

$$(1-\alpha_s)\cdot P_s \cdot q_1 \cdot \Gamma_0^{M-1} + [1-(1-\alpha_s)\cdot P_s \cdot q_1]\Gamma_1^{M-1}$$

The expressions in all the other entries are similarly argued.

We denote by V_M the value of the game Γ_1^M. Following the general procedure due to Shapley (see Section B), we obtain a difference equation for $\{V_M: M = 1,2,...,\}$ by the following method:

(1) Substitute V_{M-1} for Γ_1^{M-1} wherever it appears in the matrix of Γ_1^M. Substitute also M for Γ_0^M.

(2) Calculate the value of the resulting matrix. This value is equal to the value of Γ_1^M.

Thus, we have the equation:

$$V_M = \mathrm{val}\begin{pmatrix} q_1 + V_{M-1} & p_s q_1 (M-1) + (1-p_s q_1) V_{M-1} \\ \alpha_p q_1 + (1-\alpha_p) q_2 + V_{M-1} & (1-\alpha_s) q_1 p_s (M-1) + [1-(1-\alpha_s) q_1 p_s] V_{M-1} \\ \alpha_p q_2 + (1-\alpha_p) q_1 + V_{M-1} & \alpha_s q_1 p_s (M-1) + [1-\alpha_s q_1 p_s] V_{M-1} \\ q_2 + V_{M-1} & V_{M-1} \end{pmatrix} .$$

Since one player (the attacker) has only two pure actions, it is known that optimal randomized defensive policies exist which mix at most two of the four response programs available to the defender. It has to be emphasized that we are not interested in investigating all optimal strategies for the game, since our main interest is in the values of the games. Also, in order to make optimal strategies more likely to be ever implemented, we seek for the simplest optimal strategies, where simplest naturally means randomizing over the minimum possible number of pure actions. In our case the number is two.

We start solving the above equation by making some preliminary observations on the matrix.

First we notice that

$$V_{M-1} < M-1.$$

This is obvious since M-1 is the number of missiles the attacker has in the game Γ_1^{M-1}, so that the optimal number of penetrating missiles cannot exceed what he has, and clearly should be strictly less.

If we use this relation together with the relation $q_1 < q_2$, we can quite simply observe that no row of the above matrix dominates any other row. However, we can show that either row 2 or row 3 (which correspond to response programs P_1-S_2 and P_2-S_1, respectively) is dominated by a mixture* of rows 1 and 4. Which row is the dominated one depends upon the parameters α_p, α_s, as we now show.

First, let us compare the mixture $\alpha_p(1) + (1-\alpha_p)(4)$ with row 2. The first element in the mixture, that is, the payoff induced by the mixture when the attacker chooses the AP decision, is:

$$\alpha_p(q_1+V_{M-1}) + (1-\alpha_p)(q_2+V_{M-1}) = \alpha_p \cdot q_1 + (1-\alpha_p)q_2 + V_{M-1}$$

---

*A "mixture" will be referred to as a randomized policy, mixing two rows. We shall denote by $\delta(i) + (1-\delta)(j)$ a mixture which chooses row i with probability δ, and row j with probability $1-\delta$. Later on we shall also abbreviate and write $H_\delta^{i,j}$ for such a mixture.

which is identical to the first element of row 2. The second element of the mixture (attacker chooses AS action) is:

$$\alpha_p[P_s \cdot q_1(M-1)+(1-P_s \cdot q_1) \cdot V_{M-1}] + (1-\alpha_p) \cdot V_{M-1}$$

$$= \alpha_p \cdot P_s \cdot q_1(M-1) + (1-\alpha_p \cdot P_s \cdot q_1)V_{M-1}$$

If we compare it to the corresponding element in row 2 (i.e., the element in column 2, row 2), we see that in order for the second element of the mixture to be less than the second element of row 2, it is necessary and sufficient that:

$$\alpha_p \leq 1 - \alpha_s.$$

Since we showed that the first element (column 1) of the mixture is equal to the first element of row 2, it turns out that the above condition guarantees that row 2 is dominated by this special mixture of rows 1 and 4 ($\alpha_p(1)+(1-\alpha_p)(4)$). Hence, if $\alpha_p \leq 1-\alpha_s$, we can ignore row 2 without changing the value of the game.

Similarly one can show that if $\alpha_p \geq 1-\alpha_s$, row 3 is dominated by the mixture $(1-\alpha_p)(1)+\alpha_p(4)$. Therefore, in all cases, it is always possible to reduce the size of the matrix of the game Γ_1^M by ignoring either row 2 or row 3.

Worth noticing is the case $\alpha_p = 1-\alpha_s$. In this case, both row 2 and row 3 are dominated by mixtures of rows 1 and 4, and so we can leave inside the game only rows 1 and 4.

The various domination cases described above have a very interesting operational interpretation. To see this, notice

that $1-\alpha_s$ is the probability of classifying an anti-secondary missile as anti-primary missile. If $\alpha_p = 1-\alpha_s$, then classification probabilities are the same for both missile types, or in other words, the two types are indistinguishable. Indistinguishability makes it unnecessary for the defender to classify. He needs only to decide on some mixture of mode 1 and mode 2, regardless of the classification. A mixture of reponse programs P_1-S_1 and P_2-S_2 (rows 1 and 4) is just that kind of mixture, since both programs dictate a given decision without referring to the classification. In the cases $\alpha_p < 1-\alpha_s$ and $\alpha_p > 1-\alpha_s$ it is possible to distinguish between anti-secondary and anti-primary missiles.

The fact that the two missile types are distinguishable should be exploited in every optimal strategy. This rather intuitive perception can be formalized by saying that if $\alpha_p \neq 1-\alpha_s$, no optimal strategy exists which does not associate a positive weight with either the P_1-S_2 or the P_2-S_1 response programs. This is intuitively plausible, since the programs P_1-S_2 and P_2-S_1 are the only programs which respond differently to different classifications. In other words, these programs are the ones which exploit the distinguishability property. We present this conclusions as a theorem:

<u>THEOREM 1</u>: If $\alpha_p \neq 1-\alpha_s$, then no optimal defensive strategy exists in which P_1-S_1 and P_2-S_2 are the only active actions. If $\alpha_p > 1-\alpha_s$, then the P_1-S_2 program should be active in every mixed optimal strategy, and if $\alpha_p < 1-\alpha_s$, the P_2-S_1 program should be active in every mixed optimal strategy.

Proof: We examine the case $\alpha_p > 1-\alpha_s$ only. The other case is similar. From the previous discussion we know that in this case row 3 (P_2-S_1) can be ignored. That is, there are optimal strategies which do not involve the P_2-S_1 response program. The matrix of the game Γ_1^M is thus:

attacker / defender	AP	AS
P_1-S_1	$q_1 + V_{M-1}$	$P_s q_1 (M-1) + (1-P_s q_1) V_{M-1}$
P_1-S_2	$\alpha_p q_1 + (1-\alpha_p) q_2 + V_{M-1}$	$(1-\alpha_s) q_1 P_s (M-1) + [1-(1-\alpha_s) q_1 P_s] V_{M-1}$
P_2-S_2	$q_2 + V_{M-1}$	V_{M-1}

Now let $\delta(P_1\text{-}S_1) + (1-\delta)(P_2\text{-}S_2)$ be a mixture of rows 1 and 3 above, where δ is the weight of P_1-S_1 $(0 < \delta < 1)$. We show that this mixture cannot be an optimal strategy. First we write down the payoffs of this mixed strategy under each of the possible actions of the attacker. We shall denote the above mixture by $H_\delta^{1,3}$ and write $\text{Pay}(H_\delta^{1,3}|AP)$ and $\text{Pay}(H_\delta^{1,3}|AS)$ for the payoffs of this mixture under AP and AS decisions. We have:

$$\text{Pay}(H_\delta^{1,3}|AP) = \delta q_1 + (1-\delta) q_2 + V_{M-1} \qquad \text{(IV.5)}$$

$$\text{Pay}(H_\delta^{1,3}|AS) = \delta P_s q_1 [M-1-V_{M-1}] + V_{M-1} \qquad \text{(IV.6)}$$

We have to distinguish, as we shall see, between two possibilities concerning the value of δ: (1) $\delta > 1-\alpha_s$ and (2) $\delta \leq 1-\alpha_s$. We start from case (1). Let $H_\gamma^{1,2}$ be the mixture $\gamma(P_1-S_1) + (1-\gamma)(P_1-S_2)$, and let us calculate the two payoffs of this strategy:

$$\mathrm{Pay}(H_\gamma^{1,2}|AP) = \gamma q_1+(1-\gamma)\alpha_p q_1+(1-\gamma)(1-\alpha_p)q_2+V_{M-1} \qquad \text{(IV.7)}$$

$$\mathrm{Pay}(H_\gamma^{1,2}|AS) = [\gamma P_s q_1+(1-\gamma)(1-\alpha_s)q_1 P_s][M-1-V_{M-1}] + V_{M-1} . \qquad \text{(IV.8)}$$

If $H_\gamma^{1,2}$ is to be a preferable strategy for the defender, we must have:

$$\mathrm{Pay}(H_\gamma^{1,2}|AP) < \mathrm{Pay}(H_\delta^{1,3}|AP) \qquad \text{(IV.9a)}$$

$$\mathrm{Pay}(H_\gamma^{1,2}|AS) < \mathrm{Pay}(H_\delta^{1,3}|AS) . \qquad \text{(IV.9b)}$$

Using Eqs. (IV.5-IV.8) above, we find that these conditions are equivalent to:

$$\gamma q_1 + (1-\gamma)\alpha_p q_1 + (1-\gamma)(1-\alpha_p)q_2 < \delta q_1 + (1-\delta)q_2 \qquad \text{(IV.9c)}$$

$$\gamma P_s q_1 + (1-\gamma)(1-\alpha_s)q_1 P_s < \delta P_s q_1 \qquad \text{(IV.9d)}$$

From (IV.9c) we get

$$\gamma > \frac{\delta - \alpha_p}{1 - \alpha_p} \qquad \text{(IV.10a)}$$

and from (IV.9d) we have

$$\gamma < \frac{\delta - (1-\alpha_s)}{\alpha_s} . \qquad \text{(IV.10b)}$$

To show that there exists a value of γ which satisfies these two conditions, it is necessary that

$$\frac{\delta - \alpha_p}{1 - \alpha_p} < \frac{\delta - (1-\alpha_s)}{\alpha_s} .$$

This last inequality is equivalent to:

$$[\alpha_p - (1-\alpha_s)]\delta < \alpha_p - (1-\alpha_s),$$

and this inequality indeed holds true, since we are assuming that $\alpha_p > 1-\alpha_s$, and obviously, $\delta < 1$. Thus we can find a value of γ satisfying (IV.10a) and (IV.10b) simultaneously, and hence (IV.9a) and (IV.9b). Since we also assume that $\delta > 1-\alpha_s$, we see from Eq. (IV.10b) that it is also possible to find such a value of γ which is between 0 and 1. Thus we have proven that a strategy $H_\gamma^{1,2}$, with such a value of γ is preferable to $H_\delta^{1,3}$.

The second case for which $\delta \leq 1-\alpha_s$ is resolved in a similar way by considering a general mixture $H_\gamma^{2,3}$ (instead of $H_\gamma^{1,2}$ above). We have:

$$\text{Pay}(H_\gamma^{2,3}|\text{AP}) = \gamma[\alpha_p q_1+(1-\alpha_p)q_2] + (1-\gamma)q_2 + V_{M-1} \qquad \text{(IV.11)}$$

$$\text{Pay}(H_\gamma^{2,3}|\text{AS}) = \gamma(1-\alpha_s)P_s q_1[M-1-V_{M-1}] + V_{M-1} . \qquad \text{(IV.12)}$$

The requirements $\mathrm{Pay}(H_{\gamma}^{2,3}|AP) < \mathrm{Pay}(H_{\delta}^{1,3}|AP)$ and $\mathrm{Pay}(H_{\gamma}^{2,3}|AS) < \mathrm{Pay}(H_{\delta}^{1,3}|AS)$ are equivalent to:

$$\gamma > \frac{\delta}{\alpha_p} \qquad \text{(IV.13a)}$$

and

$$\gamma < \frac{\gamma}{1-\alpha_s} . \qquad \text{(IV.13b)}$$

Since we are assuming $\delta < 1-\alpha_s$ in this case, and since $\alpha_p > 1-\alpha_s$, it is again possible to find a value of γ, which is between 0 and 1, and which satisfies both Eqs. (IV.13a) and (IV.13b). Hence $H_{\gamma}^{2,3}$ is preferable to $H_{\delta}^{1,3}$.

We have thus proven that we can always find a defensive policy better than any policy of the type $H_{\delta}^{1,3}$. The proof of the theorem is thus complete.

Another interesting observation can be made, regarding the domination relationship between the two columns of the matrix of the game. We see that for $M = 1$, the second column is zero (since obviously $V_0 = 0$) and the first is positive. This is quite expected, since if only one missile is to be launched, the optimal decision is clearly to launch an anti-primary missile. In this case therefore, the optimal strategy for the defender is a pure one--to use the P_1-S_1 program (i.e., to use the more-effective mode).

It is expected that this domination relation will continue to hold true for some more values of M, that is, for $M = 2,3,\ldots$ until for some M, domination will no longer hold. At this

first M for which the first column (corresponding to the AP decision) ceases to dominate, both players would resort to randomized strategies for optimal behavior. We can deduce, using a remarkably simple heuristic argument, that the greatest value of M for which the solution of the game consists of a pair of <u>pure</u> strategies (AP for the attacker and P_1-S_1 for the defender) is precisely the number M* which has been calculated in Chapter II for the one-sided dynamic programming model.

The argument is the following: Suppose that for some value, which is less than or equal to the M* given in Chapter II, the optimal strategy for the defender is randomized, i.e., consists of at least one active response program in which there is positive probability of using mode 2. In the model presented in Chapter II it was assumed that the defender always operates on mode 1, and even though the attacker knew it, his optimal decision (for $M \leq M^*$) was shown to be the AP decision. Knowing now that with some probability, the defender will use mode 2 instead of mode 1 should not lead the attacker to change his course. He still should desire to launch anti-primary missiles, since in mode 2 there is a greater probability of survival and there is a zero probability of killing the secondary target anyway. Therefore, if $\tilde{M}$ is the greatest M for which the game has a saddle point in the P_1-S_1--AP entry, then the foregoing argument shows that:

$$\tilde{M} \geq M^*$$

where M* was shown in Chapter II to be given by:

$$M^* = 1 + \left\lfloor \frac{1}{P_s(1-q_1)} \right\rfloor$$

(Notice that we should use q_1, not q_2, in place of q in Eq. (II.14).)

We now have to show that $\breve{M}$ can't be greater than M*. Suppose therefore that $\breve{M} > M^*$. It would then mean that for some $M > M^*$, the optimal policy for the attacker is AP and for the defender is P_1-S_1. But then, the attacker, <u>knowing</u> that the defender should use the pure strategy P_1-S_1, would tend to aim his missiles at the secondary target, because for all $M > M^*$ this is what's preferable for him, as the one-sided model of Chapter II indicates. Thus we arrive at a contradiction to a fundamental property of equilibrium of solutions of zero-sum games. By that property, the knowledge by any player, of the fact that his opponent does use his optimal strategy, by no means attracts him to change his own policy. This is clearly not so in the above case. Thus, for no value of M greater than M* can there be a pure solution to the game. Therefore for all $M > M^*$, the optimal strategy for the attacker is mixed (the two possible decisions, AP and AS, are active). Obviously there is no M for which the optimal strategy of the attacker is pure anti-secondary because the optimal strategy of the defender would then be P_2-S_2, and the pair of pure strategies P_2-S_2--AS clearly cannot be an equilibrium combination.

We proceed now by solving the equation (which corresponds to the case $\alpha_p > 1-\alpha_s$):

$$V_M = \text{val}\begin{pmatrix} q_1 + V_{M-1} & P_s q_1 (M-1) + (1-P_s q_1) V_{M-1} \\ \alpha_p q_1 + (1-\alpha_p) q_2 + V_{M-1} & (1-\alpha_s) q_1 P_s (M-1) + [1-(1-\alpha_s) q_1 P_s] V_{M-1} \\ q_2 + V_{M-1} & V_{M-1} \end{pmatrix} \tag{IV.14}$$

From the discussion made above we know that for $M = 1,2,\ldots,M^*$ we have:

$$V_M = M \cdot q_1$$

For $M > M^*$ we know, from the Theorem 1 proven above, that the optimal defensive strategy either mixes rows 1 and 2 in the above matrix, or rows 2 and 3 (but cannot mix rows 1,3). Starting from $M = M^*+1$, we should find for each M, the two values V'_M, V''_M defined by:

$$\begin{array}{ll} V'_M = \text{val}\begin{pmatrix} q_1 + V_{M-1} & P_s q_1 (M-1) + (1-P_s q_1) V_{M-1} \\ \alpha_p q_1 + (1-\alpha_p) q_2 + V_{M-1} & (1-\alpha_s) q_1 P_s (M-1) + [1-(1-\alpha_s) q_1 P_s] V_{M-1} \end{pmatrix} & \begin{array}{l} \text{(row 1)} \\ \text{(row 2)} \end{array} \end{array} \tag{IV.15a}$$

$$\begin{array}{ll} V''_M = \text{val}\begin{pmatrix} \alpha_p q_1 + (1-\alpha_p) q_2 + V_{M-1} & (1-\alpha_s) q_1 P_s (M-1) + [1-(1-\alpha_s) q_1 P_s] V_{M-1} \\ q_2 + V_{M-1} & V_{M-1} \end{pmatrix} & \begin{array}{l} \text{(row 2)} \\ \text{(row 3)} \end{array} \end{array} \tag{IV.15b}$$

With the assumption that V_{M-1} is known (for $M = M^*+1$ we have $V_{M^*} = M^*\cdot q_1$ and the V_M's are then calculated recursively), we have to find both V'_M and V''_M, and then take the minimum in order to get V_M (because the defender, who is the one to choose the row, wishes to minimize the number of penetrators). Thus,

$$V_M = \min[V'_M, V''_M] .$$

We now simplify Eqs. (IV.15a) and (IV.15b). Define:

$$\alpha_p\cdot q_1 + (1-\alpha_p)\cdot q_2 = q$$

and

$$M - V_M = g_M$$

We multiply each of the equations (IV.15a) and (IV.15b) by -1 (the value of the matrix is thus also multiplied by -1), then add M to both sides of each equation. We write g'_M for $M - V'_M$, and g''_M for $M - V''_M$. We also make use of yet another trivial identity: Let b ben any constant, and let B be the matrix of a zero-sum game. Let also J be a matrix of the same dimensions as B, with all its entries equal to 1.

Then:

$$b + \mathrm{val}(B) = \mathrm{val}(B + bJ)$$

Using this and the above definitions, Equation (IV.15a) becomes:

$$g_M' = 1 + \text{val} \begin{pmatrix} g_{M-1} - q_1 & (1-P_s q_1) g_{M-1} \\ g_{M-1} - q & [1-(1-\alpha_s) q_1 P_s] g_{M-1} \end{pmatrix} \quad \text{(IV.16a)}$$

and Eq. (IV.15b) becomes:

$$g_M'' = 1 + \text{val} \begin{pmatrix} g_{M-1} - q & [1-(1-\alpha_s) q_1 P_s] g_{M-1} \\ g_{M-1} - q_2 & g_{M-1} \end{pmatrix} . \quad \text{(IV.16b)}$$

Equations (IV.16a) and (IV.16b) have forms which make it much more convenient to apply the known explicit formulae for the value of a 2×2 matrix (see the Appendix for these formulae). Notice also that we have transformed V_M to g_M, where g_M simply expresses the difference between the actual achievable number of penetrating missiles and the ideally desirable number (which is M). The function g_M thus measures the effect of the defense, and so serves as a very meaningful function for itself, besides its being a more convenient quantity mathematically, as we shall see.

From (IV.16a) we get, after applying formula (1) of the Appendix:

$$g_M' = 1 + \frac{(q-q_1)(1-q_1 P_s) - \alpha_s q_1^2 P_s + \alpha_s q_1 P_s g_{M-1}}{q - q_1 + \alpha_s q_1 P_s g_{M-1}} \cdot g_{M-1} .$$

We now define:

$$b' = \frac{q - q_1}{\alpha_s \cdot q_1 \cdot P_s}$$

$$a' = \frac{(q-q_1)(1-q_1P_s) - \alpha_s \cdot q_1^2 P_s}{\alpha_s \cdot q_1 \cdot P_s} ,$$

so that we have:

$$g'_M = 1 + \left(\frac{a' + g_{M-1}}{b' + g_{M-1}}\right) \cdot g_{M-1} .$$

Similarly, from Eq. (IV.16b) we can explic

value on the right side as:

$$g''_M = 1 + \frac{q_2-q-(1-\alpha_s)q_1P_sq_2+(1-\alpha_s)q_1P_sg_M}{q_2-q+(1-\alpha_s)q_1P_sg_{M-1}}$$

Define now

$$a'' = \frac{q_2 - q - (1-\alpha_s)q_1q_2P_s}{(1-\alpha_s)q_1P_s}$$

$$b'' = \frac{q_2-q}{(1-\alpha_s)q_1P_s}$$

and so we have

$$g''_M = 1 + \left(\frac{a'' + g_{M-1}}{b'' + g_{M-1}}\right) \cdot g_{M-1} .$$

Now g_M can be written as follows:

$$g_M = M-V_M = M-\min[V'_n, V''_M] = \mathrm{Max}[g'_M$$

Making the comparison between g'_M and g''_M, us

and (IV.17b), we find:

$$(g'_M \geq g''_M) \Longleftrightarrow \left(\frac{a' + g_{M-1}}{b' + g_{M-1}}\right) \geq \frac{a'' + g_{M-1}}{b'' + g_{M-1}}$$

Notice that $q_1 < q < q_2$, and so b' and
are both positive. Since g_{M-1} must al
multiply the last inequality by the tw
two sides being assured that the inequa
at the same direction. After a simple
we find:

$$(g_M' \geq g_M'') \Longleftrightarrow g_{M-1} \leq \frac{a'b'' -}{(a''-a') -}$$

We have thus found a very convenient way
optimal defensive strategy, at state M, c
programs P_1-S_1 and P_1-S_2, or rather the p
P_1-S_2: If Eq. (IV.18) is satisfied for t
the first case which holds. If not, then
Notice from (IV.18) that we have found a n
which separates the two possible structure
defense strategy. The threshold is

$$\frac{a'b'' - a''b'}{(a''-a') - (b''-b')} = \text{Val} \begin{pmatrix} -a' \\ 'b' \end{pmatrix}$$

$$= \lambda\text{, say.}$$

We now prove the following lemma which will
gaining some insight into the structure of t
sive and offensive strategies.

<u>Lemma</u>: The function g_M is an increasing f

Proof: For $M \leq M^*$, we have $V_M = M \cdot q_1$, and hence $g(M) = M - V_M = M(1-q_1)$ which is, of course, an increasing function of M. To show that this is also true for $M > M^*$, let us develop Eq. (IV.17a) as follows:

$$g'_M = 1 + \left(\frac{a' + g_{M-1}}{b' + g_{M-1}}\right) \cdot g_{M-1} = 1 + \left(1 - \frac{b' - a'}{b' + g_{M-1}}\right) \cdot g_{M-1}$$

$$= g_{M-1} + 1 - (b'-a') \cdot \frac{g_{M-1}}{b'+g_{M-1}} > g_{M-1} + 1 - (b'-a') \qquad \text{(IV.19)}$$

By definitions of a' and b' we have:

$$b' - a' = \frac{(q-q_1)q_1 P_s + \alpha_s q_1^2 P_s}{\alpha_s q_1 P_s} = \frac{q - q_1}{\alpha_s} + q_1$$

Now notice that:

$$q - q_1 = \alpha_p q_1 + (1-\alpha_p) q_2 - q_1 = (1-\alpha_p)(q_2 - q_1)$$

and so:

$$b' - a' = \left(\frac{1-\alpha_p}{\alpha_s}\right)(q_2 - q_1) + q_1 .$$

Since we work here on a case for which $\alpha_p > 1-\alpha_s$, we find from the last equality that

$$b' - a' < q_2$$

and so, returning now to Eq. (IV.19) above we find

$$g'_M > g_{M-1} + 1 - q_2 > g_{M-1}$$

Hence

$$g_M = \text{Max}[g'_M, g''_M] > g'_M > g_{M-1} .$$

This proves the lemma.

This lemma leads to a very useful conclusion about the structure of the defense optimal solution. We know from inequality (IV.18) that the least M for which the optimal strategy mixes the P_1-S_2 and P_2-S_2 response programs, can be written as M**+1, where M** is the least integer M (greater than or equal to M*) which exceeds λ:

$$M^{**} = \text{Min}\{M: \; M \geq M^*, g_M > \lambda = \frac{a'b''-a''b'}{(a''-a')-(b''-b')}\} .$$

(Notice that if $g_{M^*} = M^*(1-q_1) > \lambda$, then we have M** = M*.) Since we have just proven that g_M is an increasing function of M, then the inequality $g_M > \lambda$ will continue to hold for all values of M greater than M**. We may therefore conclude that once the optimal defense strategy switches from P_2-S_2--P_1-S_2 mixture type, to a P_1-S_1--P_1-S_2 mixture type (and this happens exactly at M = M**), it can never again switch back, at lower values of M, to a policy in which the P_2-S_2 response program is active.

In the following table we present a description of the general structure of the defensive and offensive optimal strategies.

Table IV.1: Optimal Strategies In the ASAPA Game with MENP Payoff

Number of offensive missiles left	Optimal Defensive Strategy	Optimal Offensive Strategy
$M \leq M^*$	Pure P_1-S_1 response program.	Pure AP policy.
$M^* < M \leq M^{**}$ (*)	Randomization over P_1-S_1 and P_1-S_2 response programs.	Randomization over AP and AS decisions.
$M > M^{**}$	Randomization over P_2-S_2 and P_1-S_2 response programs.	Randomization over AP and AS decisions.

(*) If $M^{**} = M^*$, then the second row of the table is vacuous.

The interpretation of the structure presented in Table IV.1 is this: When there are only a few missiles left to be launched ("few" means less than or equal to M*), the defender's main concern is the prevention of missile penetration--not so much his own survival, and so he must use mode 1 only.

If the number of missiles left is large enough (more than M**), he must be more concerned about his survival than about preventing penetration of immediate missiles. This is reflected by the relatively frequent use he should make of mode 2, which renders the secondary target less vulnerable, although also less effective. The "emphasis" on mode 2 is shown by the

fact that he mixes the P_1-S_2 and P_2-S_2 response programs. (He never uses mode 1 "blindly".)

There is also an interim zone (if $M^{**} > M^*$) between M^* and M^{**}, where the defender must be cautious. At that zone he must evaluate the interception of potential anti-primary missiles as more important than his own protection against potential anti-secondary missiles. Thus he randomizes over P_1-S_1 and P_1-S_2.

To conclude the analysis of this model, we calculate the optimal strategies of both the defender and the attacker, in each of the three zones of M described above.

(a) $\underline{M \leq M^*}$: Both players use pure strategies. The attacker launches anti-primary missiles, and the defender uses P_1-S_1 response program.

(b) $\underline{M^* < M \leq M^{**}}$: (If $M^{**} = M^*$, which is possible, this zone is empty.) We denote by $\delta_{A*}(M)$ the probability of the attacker using an anti-primary missile (at his optimal strategy) and by $\delta_{D*}(M)$ the defender using the P_1-S_1 response program at his optimal strategy. ($1-\delta_{D*}(M)$ is the probability of using P_1-S_2 response program.) The calculation of $\delta_{D*}(M)$ and $\delta_{A*}(M)$ is straightforward, based on the formulae (2) and (3) in the Appendix. We first write the active sub-matrix of the game at state M (for $M^* < M \leq M^{**}$):

$$
\begin{array}{cc}
 & \text{Attacker} \\
 & \begin{array}{cc} \text{AP} & \text{AS} \end{array} \\
\text{Defender} \begin{array}{c} P_1\text{-}S_1 \\ \\ P_1\text{-}S_2 \end{array} & \begin{pmatrix} g_{M-1}-q_1 & (1-P_s q_1)\cdot g_{M-1} \\ & \\ g_{M-1}-q & [1-(1-\alpha_s)q_1P_s]g_{M-1} \end{pmatrix}
\end{array}
$$

Using formulae (2),(3) in the Appendix we find:

$$
\delta_{D*}(M) = \frac{[1-(1-\alpha_s)q_1P_s]g_{M-1} - (g_{M-1}-q)}{q-q_1 + \alpha_s q_1 P_s g_{M-1}}
$$

$$
= 1 - \frac{q_1(P_s\cdot g_{M-1} - 1)}{q-q_1 + \alpha_s\cdot q_1 P_s\cdot g_{M-1}} \qquad \text{(IV.20a)}
$$

$$
\delta_{A*}(M) = \frac{[1-(1-\alpha_s)q_1P_s]g_{M-1} - (1-P_s q_1)\cdot g_{M-1}}{q-q_1 + \alpha_s\cdot q_1\cdot P_s\cdot g_{M-1}}
$$

$$
= \frac{1}{\dfrac{q-q_1}{\alpha_s q_1 P_s} + g_{M-1}} = \frac{1}{b' + g_{M-1}} \qquad \text{(IV.20b)}
$$

Notice that $g_{M*} = M^* - V_{M*} = M^*-M^*q_1 = M^*(1-q_1)$ and since

$$
M^* > \frac{1}{P_s(1-q_1)}
$$

we conclude that $g_{M*} > \frac{1}{P_s}$. Since g_M is monotone increasing function of M, we have, for all $M > M^*$, $g_M > \frac{1}{P_s}$. From this one can easily show that Eqs. (IV.20a) and (IV.20b) yield values of $\delta_{D*}(M)$ and $\delta_{A*}(M)$ between 0 and 1, as they should.

(c) $M > M^{**}$: In this zone, the active submatrix is:

$$
\begin{array}{lll}
 & & \text{Attacker} \\
 & & \begin{array}{cc} \text{AP} & \text{AS} \end{array} \\
\text{Defender} & \begin{array}{l} P_1\text{-}S_2 \\ \\ P_2\text{-}S_2 \end{array} & \begin{pmatrix} g_{M-1}-q & [1-(1-\alpha_s)q_1P_s]g_{M-1} \\ & \\ g_{M-1}-q_2 & g_{M-1} \end{pmatrix}
\end{array}
$$

We denote here by $\delta_{D**}(M)$ ($\delta_{A**}(M)$) the weight of the P_2-S_2 response program in the optimal defensive strategy (weight of AP action in the optimal attacker's strategy). We again apply formulae (2), (3) of the Appendix, and find that:

$$\delta_{D**}(M) = \frac{(1-\alpha_s)q_1P_sg_{M-1}-q}{q_2-q+(1-\alpha_s)q_1P_sg_{M-1}} \tag{IV.21a}$$

$$\delta_{A**}(M) = \frac{(1-\alpha_s)q_1P_s\cdot g_{M-1}}{q_2-q+(1-\alpha_s)q_1P_s\cdot g_{M-1}} = \frac{1}{b''+g_{M-1}} \tag{IV.21b}$$

It can be seen that $g_M \to \infty$ as $M \to \infty$ (since $g_M > g_{M-1} + (1-q_2)$, as was shown in the proof of the lemma). Thus, we find that:

$$\delta_{D**}(M) \to 1 \quad \text{and} \quad \delta_{A**}(M) \to 0 \quad \text{as} \quad M \to \infty.$$

The solution of the ASAPA game model, with MENP payoff is now completely at hand. For convenience of use we summarize here the alogrithm for solving the game.

Algorithm--ASAPA Game Model, With MENP Criterion:

(1) Calculate M* (Eq. (14), Chapter II). Calculate the the constants a', a", b', b" from the parameters of the problem.

(2) For all values of M less than or equal to M^*, put:

$$V_M = M \cdot q_1$$

The optimal strategies are: Pure AP for the attacker, and pure P_1-S_1 for the defender.

(3) Set $M = M^*+1$. Calculate

$$g_{M^*} = M^* - V_{M^*}$$

(4) Check whether

$$g_{M-1} > \lambda = \frac{a'b'' - a''b'}{(a'' - a') - (b'' - b')}$$

If this inequality doesn't hold, calculate g_M from Eq. (17a) (taking $g_M = g_M'$). Calculate V_M by $V_M = M - g_M$. Calculate optimal strategies from Eqs. (20a) and (20b). Set $M = M+1$ and repeat the step.

If the above inequality does hold true, set $M^{**} = M-1$. Set $M = M+1$.

(5) Calculate $g_M = g_M''$, given by Eq. (17b). Calculate optimal strategies using Eqs. (21a) and (21b). Set $M = M+1$ and repeat the step.

F. THE ANTI-PRIMARY/ANTI-SECONDARY ALLOCATION (ASAPA) GAME MODEL WITH MAXIMUM PROBABILITY OF HIT (MPH) CRITERION

This section differs from the previous one by the function used to define the payoff of the game. We now define the payoff of a game, Γ_1^M or Γ_0^M as the probability of eventually hitting the primary target. For reasons of mathematical

convenience we use probabilities of miss--rather than hit probabilities--in all the analyses which follow. This does not affect in any way the optimal strategies of either the defender or the attacker.

First note the obvious fact that the value of the game Γ_0^M is identical with the probability of missing the target in M attempts, each having probability of hit P_p. This is true, since by definition of Γ_0^M, the secondary target is already dead at the beginning of this game. Thus we have:

$$\mathrm{val}(\Gamma_0^M) = (1 - P_p)^M.$$

The matrix of the game Γ_1^M is:

Defender's choices \ Attacker's choices	Anti-primary Missile (AP)	Anti-secondary Missile (AS)
P_1-S_1	$(1-P_p q_1)\Gamma_1^{M-1}$	$P_s q_1 \Gamma_0^{M-1} + (1-P_s q_1)\Gamma_1^{M-1}$
P_1-S_2	$\alpha_p(1-P_p q_1)$ $+(1-\alpha_p)(1-P_p q_2)\Gamma_1^{M-1}$	$(1-\alpha_s)q_1 P_s \Gamma_0^{M-1}$ $+[1-(1-\alpha_s)q_1 P_s]\Gamma_1^{M-1}$
P_2-S_1	$\alpha_p(1-P_p q_2)$ $+(1-\alpha_p)(1-P_p q_1)\Gamma_1^{M-1}$	$\alpha_s q_1 P_s \Gamma_0^{M-1}$ $+(1-\alpha_s q_1 P_s)\Gamma_1^{M-1}$
P_2-S_2	$(1-P_p q_2)\Gamma_1^{M-1}$	Γ_1^{M-1}

We use the parameters q and $\bar{q}$ defined by:

$$q = \alpha_p q_1 + (1-\alpha_p) q_2 \qquad \bar{q} = \alpha_p q_2 + (1-\alpha_p) q_1 .$$

As before, let us denote by V_M the value of the game Γ_1^M. To find the sequence of values V_M ($M = 1,2,\ldots$), we start from the obvious relation

$$V_1 = \text{val}\ \Gamma_1^1 = 1 - P_p \cdot q_1$$

and use the recursive equation:

$$V_M = \text{val} \begin{pmatrix} (1-P_p q_1) V_{M-1} & q_1 P_s (1-P_p)^{M-1} + (1-q_1 P_s) V_{M-1} \\ (1-P_p q) V_{M-1} & (1-\alpha_s) q_1 P_s (1-P_p)^{M-1} + [1-(1-\alpha_s) q_1 P_s] V_{M-1} \\ (1-P_p \bar{q}) V_{M-1} & \alpha_s q_1 P_s (1-P_p)^{M-1} + [1-\alpha_s q_1 P_s] V_{M-1} \\ (1-P_p q_2) V_{M-1} & V_{M-1} \end{pmatrix} . \tag{IV.22}$$

The entries in the 4×2 matrix of Eq. (IV.22) represent "payments" which the column player (the attacker) pays to the row player (defender). Therefore the goal of the defender is to maximize payments. We can show that if $\alpha_p \leq 1-\alpha_s$, the second row of the above matrix can be ignored and if $\alpha_p \geq 1-\alpha_s$ the third can be ignored. This is done exactly as it was done in the previous section: We multiply the first row by

α_p and the fourth row by $1-\alpha_p$ and add. This mixture is directly seen to be preferable to the second row if $\alpha_p \leq 1-\alpha_s$. If $\alpha_p \geq 1-\alpha_s$ we take the mixture $(1-\alpha_p)$ (1^{st} row) + α_p (4^{th} row) and confirm that it is better than the third row. If $\alpha_p = 1-\alpha_s$ (i.e., probability of classifying a missile as anti-primary is the same for anti-primary and anti-secondary missile), then both the second and third rows can be ignored, and an optimal policy which uses only the P_1-S_1 and P_2-S_2 response programs can be found.

We see that in no case do we need to consider all the four possible response programs. Only three (at most) should be considered. From here on we solve in detail the case $\alpha_p > 1-\alpha_s$ only. The case $\alpha_p < 1-\alpha_s$ is similar and could be carried out exactly the same way as we shall do here (with q replacing $\bar{q}$, and $1-\alpha_s$ replacing α_s).

As we saw, since $\alpha_p > 1-\alpha_s$, we can ignore the third row. We now introduce the function h_M defined by:

$$h_M = \frac{V_M}{(1-P_p)^M} . \qquad \text{(IV.23)}$$

Notice that h_M measures the ratio between the actual probability of miss, and the probability of miss that would have existed had the secondary target already been destroyed. Clearly, $h_M \geq 1$. Dividing now Eq. (IV.22) by $(1-P_p)^M$, ignoring the third row as explained, and using definition (IV.23), we reach the following equation:

$$h_M = \frac{1}{1-P_p} \cdot \mathrm{val} \begin{pmatrix} (1-P_p q_1) h_{M-1} & q_1 P_s + (1-q_1 P_s) h_{M-1} \\ (1-P_p q) h_{M-1} & (1-\alpha_s) q_1 P_s + [1-(1-\alpha_s) q_1 P_s] h_{M-1} \\ (1-P_p q_2) h_{M-1} & h_{M-1} \end{pmatrix}$$

(IV.24)

We show that no optimal policy can mix the first and third rows only. This result was proven in Theorem 1 for the ASAPA model with MENP criterion. As before (Section E) we denote by $H_\delta^{i,j}$ the randomized defensive policy in which the i^{th} row (of the matrix given in Eq. (IV.24)) is selected with probability δ and j^{th} row is selected with probability $1-\delta$.

<u>Theorem 2</u>: A randomized defensive policy $H_\delta^{1,3}$ cannot be optimal, for any δ such that $0 < \delta < 1$.

The method of the proof is identical with that which was used to prove the same theorem in the previous section. Different algebraic expressions are involved, however, in this case, since the matrix of the game Γ_1^M examined in this section differs from the matrix which was treated in Section E.

<u>Proof</u>: We wish to prove that no policy of type $H_\delta^{1,3}$ can be optimal. We first write the payoffs associated with the policy $H_\delta^{1,3}$, which correspond to the two possible attacker's actions. These payoffs are calculated directly from the matrix which appears in Eq. (IV.24).

$$\mathrm{Pay}(H_{\delta}^{1,3} | AP) = [1 - P_p(\delta q_1 + (1-\delta) q_2)] h_{M-1} \qquad \text{(IV.25a)}$$

$$\mathrm{Pay}(H_{\delta}^{1,3} | AS) = h_{M-1} - \delta q_1 \cdot P_s [h_{M-1} - 1] . \qquad \text{(IV.25b)}$$

We now examine separately two cases: (1) $\delta > 1-\alpha_s$ and (2) $\delta \leq 1-\alpha_s$. We show that in case (1), a policy of the type $H_{\gamma}^{1,2}$ can always be found which is preferable on the policy $H_{\delta}^{1,3}$, and in case (2) a policy $H_{\gamma}^{2,3}$ exists which is preferable.

We begin with the case $\delta > 1-\alpha_s$. We explicitly write the payoffs of a policy $H_{\gamma}^{1,2}$ for arbitrary γ:

$$\mathrm{Pay}(H_{\gamma}^{1,2} | AP) = [1 - P_p(\gamma q_1 + (1-\gamma) q)] h_{M-1} \qquad \text{(IV.26a)}$$

$$\mathrm{Pay}(H_{\gamma}^{1,2} | AS) = h_{M-1} - q_1 P_s [\gamma + (1-\gamma)(1-\alpha_s)][h_{M-1} - 1] . \qquad \text{(IV.26b)}$$

It is obvious that $H_{\gamma}^{1,2}$ is preferable on $H_{\delta}^{1,3}$ if and only if:

$$\mathrm{Pay}(H_{\gamma}^{1,2} | AP) > \mathrm{Pay}(H_{\delta}^{1,3} | AP) \qquad \text{(IV.27a)}$$

and

$$\mathrm{Pay}(H_{\gamma}^{1,2} | AS) > \mathrm{Pay}(H_{\delta}^{1,3} | AS) . \qquad \text{(IV.27b)}$$

By comparing Eqs. (IV.25a) and (IV.26a), and then Eqs. (IV.25b) and (IV.26b), we find that conditions (IV.27a) and (IV.27b) are equivalent to the conditions:

$$\gamma q_1 + (1-\gamma) q < \delta q_1 + (1-\delta) q_2$$

and

$$\gamma + (1-\gamma)(1-\alpha_s) < \delta \quad .$$

We now use the relation $q = \alpha_p q_1 + (1-\alpha_p) q_2$ and find that the first inequality in the last set is equivalent to

$$\gamma > \frac{\delta - \alpha_p}{1 - \alpha_p} \tag{IV.28}$$

and the second inequality is equivalent to

$$\gamma < \frac{\delta - (1-\alpha_s)}{\alpha_s} \quad . \tag{IV.29}$$

To show that there exists a value of γ (between 0 and 1) which satisfies both inequalities (IV.28) and (IV.29) we only have to show that:

$$\frac{\delta - \alpha_p}{1 - \alpha_p} < \frac{\delta - (1-\alpha_s)}{\alpha_s} \quad .$$

It is straightforward to confirm this last relation, using the relation $\alpha_p > 1-\alpha_s$, which is assumed to hold in our case. The other assumption ($\delta > 1-\alpha_s$) which has been made, guarantees that not both expressions in the right hand sides of inequalities (IV.28) and (IV.29) are either greater than 1 or less than 0, so that a value of γ, which is indeed a probability (i.e., between 0 and 1), exists that satisfies Eqs. (IV.28) and (IV.29).

We turn to the second case, i.e., $\delta \le 1-\alpha_s$. We consider an $H_\gamma^{2,3}$ policy, for arbitrary γ. The payoffs are:

$$\mathrm{Pay}(H_\gamma^{2,3} | AP) = [1-P_p(\gamma q + (1-\gamma) q_2)] h_{M-1} \tag{IV.30a}$$

and

$$\mathrm{Pay}(H_{\gamma}^{2,3}|AS) \;=\; h_{M-1} - \gamma q_1 P_s (1-\alpha_s)(h_{M-1} - 1) \;. \qquad \text{(IV.30b)}$$

We are interested in a value of γ which simultaneously satisfies:

$$\mathrm{Pay}(H_{\gamma}^{2,3}|AP) \;>\; \mathrm{Pay}(H_{\delta}^{1,3}|AP) \qquad \text{(IV.31a)}$$

and

$$\mathrm{Pay}(H_{\gamma}^{2,3}|AS) \;>\; \mathrm{Pay}(H_{\delta}^{1,3}|AS) \qquad \text{(IV.31b)}$$

Making the appropriate comparisons ((IV.30a) with (IV.31a), (IV.30b) with (IV.31b)), we deduce that the following inequalities are equivalent to (IV.31a) and (IV.31b):

$$\gamma q + (1-\gamma)q_2 \;<\; \delta q_1 + (1-\delta)q_2$$

and

$$\gamma(1-\alpha_s) \;<\; \delta$$

These conditions are further equivalent to:

$$\gamma \;>\; \frac{\delta}{\alpha_p} \qquad \text{(IV.32)}$$

and

$$\gamma \;<\; \frac{\delta}{1-\alpha_s} \;. \qquad \text{(IV.33)}$$

Since $\alpha_p > 1-\alpha_s$, and $\delta \leq 1-\alpha_s$ (in this subcase) it is immediately verified that there exists some γ, between 0 and 1,

which satisfies inequalities (IV.32) and (IV.33), simultaneously, hence also (IV.31a) and (IV.31b).

We have thus shown that a policy $H_{\delta}^{1,3}$ cannot be optimal, since whatever the value of δ might be, there exists a better policy. The proof of Theorem 2 is thus complete.

Our next step is to find the recursion relations for h_M and to calculate optimal strategies. Here again, as in the ASAPA model with MENP criterion discussed in the last section, the optimal strategies are pure for all M less than or equal to M*, where M* in this case is given by (see Eq. (II.6)):

$$M^* = 1 + \left\lfloor \frac{\ln\left(1 - \frac{P_p}{P_s}\right)}{\ln\left(\frac{1 - P_p}{1 - P_p q}\right)} \right\rfloor .$$

The argument which supports that statement is exactly that which was explained before (p. 147). The fact that we use a different payoff does not affect the validity of that argument; it only changes the value of M*.

For $M \leq M^*$, the optimal pure strategy of the attacker is AP and the optimal pure strategy of the defender is the P_1-S_1 response program. The value V_M of the game Γ_1^M, for $M \leq M^*$ is thus:

$$V_M = (1-P_p \cdot q_1)^M$$

and so

$$h_M = \left(\frac{1-P_p \cdot q_1}{1-P_p}\right)^M \quad (\text{for } M \leq M^*).$$

For M · M*, we define:

$$h'_M = \frac{1}{(1-P_p)} \text{val} \begin{pmatrix} (1-P_p q_1) h_{M-1} & q_1 P_s + (1-q_1 P_s) h_{M-1} \\ (1-P_p q) h_{M-1} & (1-\alpha_s) q_1 P_s + [1-(1-\alpha_s) q_1 P_s] h_{M-1} \end{pmatrix}$$

(IV.34)

$$h''_M = \frac{1}{(1-P_p)} \text{val} \begin{pmatrix} (1-P_p q) h_{M-1} & (1-\alpha_s) q_1 P_s + [1-(1-\alpha_s) q_1 P_s] h_{M-1} \\ (1-P_p q_2) h_{M-1} & h_{M-1} \end{pmatrix} .$$

(IV.35)

Notice that h'_M is the value of a game in which only the first and second rows of the Γ_1^M game matrix (see Eq. (IV.24)), corresponding to the P_1-S_1 and P_1-S_2 response programs (respectively), are active. Similarly, h''_M, the value when the second and third rows (P_1-S_2 and P_2-S_2 response programs) are active. Since we proved that the first and third rows cannot be the only two active rows in an optimal defensive strategy, we conclude that:

$$h_M = \text{Max}[h'_M, h''_M] .$$

(We take the maximum because the payoffs in Γ_1^M expresses the probability of miss of the primary target, which the defender is interested in maximizing. The value h_M is always proportional to V_M, with a positive constant of proportion, so that the Maximum operation is preserved.)

We calculate now h'_M and h''_M from Eqs. (IV.34) and (IV.35), using the formulae given in the Appendix. From Eq. (IV.34) we find:

$$h'_M = \frac{1}{(1-P_p)}\left[h_{M-1} - \mathrm{val}\begin{pmatrix} P_p q_1 h_{M-1} & q_1 P_s (h_{M-1}-1) \\ P_p q h_{M-1} & (1-\alpha_s) q_1 P_s (h_{M-1}) \end{pmatrix}\right]$$

$$= \frac{1}{(1-P_p)}\left(h_{M-1} + \frac{P_p P_s q_1 [\alpha_s q_1 + (1-\alpha_p)(q_2-q_1)](h_{M-1}-1) h_{M-1}}{P_p(1-\alpha_p)(q_2-2_1) h_{M-1} + \alpha_s q_1 P_s (h_{M-1}-1)}\right), \qquad \text{(IV.34a)}$$

(where we have skipped here some algebraic details). From Eq. (IV.35) we get:

$$h''_M = \frac{1}{(1-P_p)}\left[h_{M-1} - \mathrm{val}\begin{pmatrix} P_p q h_{M-1} & (1-\alpha_s) q_1 P_s (h_{M-1}-1) \\ P_p q_2 h_{M-1} & 0 \end{pmatrix}\right]$$

$$= \frac{1}{(1-P_p)}\left(h_{M-1} + \frac{(1-\alpha_s) P_p P_s q_1 q_2 h_{M-1} (h_{M-1}-1)}{P_p \alpha_p (q_2-q_1) h_{M-1} + (1-\alpha_s) q_1 P_s (h_{M-1}-1)}\right) \qquad \text{(IV.35a)}$$

We now define

$$c' = \frac{P_p(1-\alpha_p)(q_2-q_1)}{\alpha_s q_1 + (1-\alpha_p)(q_2-q_1)}$$

$$c'' = \frac{P_p \alpha_p (q_2-q_1)}{(1-\alpha_s) q_2}$$

$$d' = \frac{\alpha_s q_1 P_s}{\alpha_s q_1 + (1-\alpha_p)(q_2-q_1)}$$

$$d'' = \frac{q_1 P_s}{q_2}$$

Equations (IV.34a), (IV.35a) can be rewritten as:

$$h'_M = \frac{1}{(1-P_p)}\left[h_{M-1} + \frac{P_p P_s q_1 h_{M-1}(h_{M-1}-1)}{c' h_{M-1} + d'(h_{M-1}-1)}\right] \qquad \text{(IV.34b)}$$

$$h''_M = \frac{1}{(1-P_p)}\left[h_{M-1} + \frac{P_p P_s q_1 h_{M-1}(h_{M-1}-1)}{c'' h_{M-1} + d''(h_{M-1}-1)}\right] \qquad \text{(IV.35b)}$$

Notice that since $q_2 > q_1$, the parameters c', c", d' are all positive. Since c" > 0, we find from Eq. (IV.35b) that:

$$h''_M > \frac{1}{(1-P_p)}\left[h_{M-1} - \frac{P_p P_s q_1}{d''} h_{M-1}\right]$$

$$= \frac{1-P_p q_2}{1-P_p} h_{M-1} \geq h_{M-1} \quad \left(\text{since } \frac{1-P_p q_2}{1-P_p} \geq 1\right)$$

and so $\{h_M\}$ is shown to be a monotone increasing sequence:

$$h_M = \mathrm{Max}[h'_M, h''_M] \geq h''_M > h_{M-1} .$$

Using Equations (IV.34b) and (IV.35b) it is straightforward now to deduce the general conditions for the P_1-S_2 and P_2-S_2 response programs to be the two active decisions in the optimal defensive strategy. The condition is

$$h'_M < h''_M ,$$

which by Eqs. (IV.34b) and (IV.35b) is equivalent to:

$$c''h_{M-1} + d''(h_{M-1} - 1) < c'h_{M-1} + d'(h_{M-1} - 1) .$$

This is also equivalent to

$$(c' - c'' + d' - d'')h_{M-1} > d' - d'' . \qquad \text{(IV.36)}$$

In order to draw useful conclusions from inequality (IV.36) we first prove that

$$d' - d'' > 0$$

To accomplish this, notice that

$$d' = \frac{\alpha_s q_1 P_s}{\alpha_s q_1 + (1-\alpha_p)(q_2-q_1)}$$

$$= \frac{q_1 P_s}{q_2 - [\alpha_p q_2 - (\alpha_p - (1-\alpha_s))q_1]} .$$

Now, since $\alpha_p > 1-\alpha_s$ and $q_2 > q_1$, we have

$$\alpha_p q_2 - (\alpha_p - (1-\alpha_s))q_1 > 0 ,$$

and therefore:

$$d' > \frac{q_1 P_s}{q_2} = d'' \qquad d' - d'' > 0 ,$$

as was to be shown. Returning now to inequality (IV.36) we see that if the parameters of the problem $(P_p, P_s, q_1, q_2, \alpha_p, \alpha_s)$ are such that:

$$c' - c'' + d' - d'' \leq 0 \qquad \text{(IV.37)}$$

then inequality (IV.36) cannot hold true for any value of M. Since inequality (IV.37) was shown to be equivalent to h'_M h''_M, we conclude that condition (IV.37) is sufficient to guarantee that for all values of M (greater than M*), the optimal defensive strategy comprises the P_1-S_2 and P_2-S_2 response programs. If, on the other hand, we have

$$c' - c'' + d' - d'' > 0,$$

then, since also $d'-d'' > 0$ we see that inequality (IV.36) is equivalent to:

$$h_{M-1} > \frac{d' - d''}{c' - c'' + d' - d''} . \qquad \text{(IV.38)}$$

Inequality (IV.38) now serves as the sufficient and necessary condition for h''_M to be greater than h'_M, or, for the optimal defensive policy to randomize over P_1-S_2 and P_2-S_2 response programs.

The analogy between the ASAPA game model with the MPH criterion treated here, and the ASAPA game model with the MENP criterion discussed in Section ˜ ow becomes evident. We define here the value M** of . ry.

$$M^{**} = \mathrm{Min}\{M: \; M \geq M^*, h_M > \frac{d' - d''}{c' - c'' + d' - d''}\} .$$

From the discussion above it is clear that M**+1 is the least value of M such that the optimal defensive strategy of the game Γ_1^M consists of the P_1-S_2 and P_2-S_2 response programs.

By definition of M^{**}, and since h_M was shown to be increasing with M, it is clear that for all values of M greater than or equal to M^{**}, the inequality (IV.38) is satisfied. But this inequality itself was shown to be equivalent to an optimal strategy mixing P_1-S_2 and P_2-S_2 response programs. We thus reach a structure of optimal strategies which resembles that which was discovered for the ASAPA game with MENP criterion (see Table IV.1):

(1) For $M \leq M^*$, the optimal strategies of both defender and attacker are pure: P_1-S_1 for the defender and AP for the attacker.

(2) For $M^* < M \leq M^{**}$ (if indeed $M^{**} > M^*$. It is possible that $M^{**} = M^*$ and then this case is not possible): The optimal strategy for the defense is of P_1-S_1--P_1-S_2 type (defender never uses mode 2 "blindly"). The attacker's optimal policy is also randomized (AP-AS).

(3) For $M > M^{**}$, the optimal defensive strategy is of the P_1-S_2--P_2-S_2 type (never uses mode 1 "blindly"). The attacker's optimal policy is randomized.

The operational interpretation of this structure is the same as described in Section E.

We proceed by calculating the optimal strategies for both the defender and the attacker. As before, we denote by $\xi_{A^*}(M)$ ($\xi_{D^*}(M)$) the probability of selecting the AP action at the attacker's optimal strategy (probability of selecting the P_1-S_1

response program at the defender's optimal strategy), for values of M such that $M^* < M \leq M^{**}$. Similarly, $\delta_{A^{**}}(M)$ ($\delta_{D^{**}}(M)$) are defined for $M > M^{**}$ ($\delta_{D^{**}}(M)$ is the probability of selecting the P_2-S_2 response program by the defender). We apply formulae (2) and (3) of the Appendix to the matrices shown in Eqs. (IV.34) and (IV.35), and get the following expressions:

(1) For M between M^* and M^{**} (in case $M^{**} > M^*$):

$$\delta_{D^*}(M) = \frac{[P_p q-(1-\alpha_s)q_1 P_s]h_{M-1} + (1-\alpha_s)q_1 P_s}{P_p(1-\alpha_p)(q_2-q_1)h_{M-1} + \alpha_s q_1 P_s(h_{M-1}-1)} \tag{IV.39a}$$

$$\delta_{A^*}(M) = \frac{\alpha_s q_1 P_s[h_{M-1}-1]}{P_p(1-\alpha_p)(q_2-q_1)h_{M-1} + \alpha_s q_1 P_s(h_{M-1}-1)} \tag{IV.39b}$$

(2) For $M > M^{**}$:

$$\delta_{D^{**}}(M) = \frac{[P_p q-(1-\alpha_s)q_1 P_s]h_{M-1} + (1-\alpha_s)q_1 P_s}{P_p\alpha_s(q_2-q_1)h_{M-1} + (1-\alpha_s)q_1 P_s(h_{M-1}-1)} \tag{IV.40a}$$

$$\delta_{A^{**}}(M) = \frac{(1-\alpha_s)q_1 P_s[h_{M-1}-1]}{P_p\alpha_p(q_2-q_1)h_{M-1} + (1-\alpha_s)q_1 P_s(h_{M-1}-1)} \tag{IV.40b}$$

We now summarize the algorithm for solving the ASAPA game model with the MENP criterion.

<u>Algorithm</u>--ASAPA Game Model with MPH Criterion:

(1) Calculate M^* (Eq. (II.6)). Calculate also c', c", d', d" (functions of the parameters).

(2) For all M less than or equal to M^*, set

$$V_M = (1-P_p q_1)^M$$

The optimal strategies are: Pure AP for the attacker; pure P_1-S_1 for the defender.

(3) Set $M = M^*+1$. Calculate:

$$h_{M^*} = \frac{V_{M^*}}{(1-P_p)^{M^*}} = \left(\frac{1-P_p q_1}{1-P_p}\right)^{M^*} .$$

(4) Check whether

$$h_{M-1} > \frac{d' - d''}{c' - c'' + d' - d''}$$

If this inequality doesn't hold, calculate h_M using Eq. (IV.34b) (taking h_M equal to h_M'). Calculate V_M by $V_M = (1-P_p)^M \cdot h_M$. Calculate optimal strategies using Eqs. (IV.39a) and (IV.39b). Set $M = M+1$, and repeat the step.

If the above inequality does hold true, set $M^{**} = M-1$. Set $M = M+1$.

(5) Calculate h_M using Eq. (IV.35b) (taking $h_M = h_M''$). Calculate optimal strategies using Eqs. (IV.40a) and (IV.40b). Calculate $V_M = h_M(1-P_p)^M$. Set $M = M+1$ and repeat the step.

G. ANTI-SECONDARY/ANTI-PRIMARY ALLOCATION (ASAPA) GAME MODEL WITH MINIMUM EXPECTED COST (MEC) CRITERION

We now present the ASAPA game model using the criterion of minimal cost of destroying the primary target (MEC-criterion). In this model the stochastic game is allowed to continue until the primary target is destroyed. Clearly the probability that

the primary target will be destroyed in a finite number of stages is equal to 1. The attacker desires to destroy the target with minimal expenses. We assume that the defender aims at exactly the opposite goal, i.e., to maximize this cost, so that the model fits into the frame of a zero-sum game.

We denote by C_p the cost of an anti-primary missile and by C_s the cost of an anti-secondary missile. We use also all notations of the last two sections. The cost of destroying the primary target is the total cost of all missiles which are consumed in the game. This cost is, of course, a random variable. The expectation of that random variable is the payoff of the game. The symbol Γ^1 stands for the game played when the secondary target is alive, and Γ^o stands for the game played when the secondary target is not present. Clearly

$$\mathrm{val}(\Gamma^o) = \frac{C_p}{P_p}$$

because in the absence of the secondary target, the expected number of missiles (all anti-primary ones, obviously) that will be consumed before the primary target is hit is $1/P_p$.

In this stochastic game model we therefore have only two game elements. We now write the full matrix of the game Γ^1. The available actions to both players are the same as before, so that in general we have a 4×2 matrix. (The expressions given in the entries of the matrix are self-explanatory so that we do not give any further explanations.)

Defender's Response Programs \ Attacker's actions	Anti-primary	Anti-secondary
P_1-S_1	$C_p+(1-P_pq_1)\Gamma^1$	$C_s+P_sq_1\Gamma^0+(1-P_sq_1)\Gamma^1$
P_1-S_2	$C_p+[1-(\alpha_pP_pq_1$ $+(1-\alpha_p)P_pq_2)]\Gamma^1$	$C_s+(1-\alpha_s)P_sq_1\Gamma^0$ $+[1-(1-\alpha_s)P_sq_1]\Gamma^1$
P_2-S_1	$C_p+[1-(\alpha_pP_pq_2$ $+(1-\alpha_p)P_pq_1)]\Gamma^1$	$C_s+\alpha_sP_sq_1\Gamma^0$ $+[1-\alpha_sP_sq_1]\Gamma^1$
P_2-S_2	$C_p+(1-P_pq_2)\Gamma^1$	$C_s+\Gamma^1$

We shall again assume that $\alpha_p > 1-\alpha_s$. Under this assumption it can be shown that the third row is dominated by a mixture of the first and the fourth rows. To see this we substitute $U_o = \text{val}(\Gamma^o) = C_p/P_p$, $U_1 = \text{val}(\Gamma^1)$ for Γ^o and Γ^1 in the above matrix. Then we consider the mixture $(1-\alpha_p)(1^{st}$ row) + α_p (4^{th} row) and use the obvious relations $U_o < U_1$. Thus we ignore the third row, and the equation which must be solved to find the value U_1 is:

$$U_1 = \mathrm{val}\begin{pmatrix} C_p+(1-P_pq_1)U_1 & C_s+\dfrac{P_sq_1C_p}{P_p}+(1-P_sq_1)U_1 \\ C_p+[1-(\alpha_pP_pq_1+(1-\alpha_p)P_pq_2)]U_1 & C_s+(1-\alpha_s)P_sq_1\dfrac{C_p}{P_p}+[1-(1-\alpha_s)P_sq_1]U_1 \\ C_p+(1-P_pq_2)U_1 & C_s+U_1 \end{pmatrix}$$

$$= C_p+U_1-\mathrm{val}\begin{pmatrix} P_pq_1U_1 & C_p-C_s+P_sq_1(U_1-\dfrac{C_p}{P_p}) \\ P_pqU_1 & C_p-C_s+(1-\alpha_s)P_sq_1(U_1-\dfrac{C_p}{P_p}) \\ P_pq_2U_1 & C_p-C_s \end{pmatrix} \qquad \text{(IV.41)}$$

where we have made use of the relation $q = \alpha_pq_1 + (1-\alpha_p)q_2$ (in the (2,1) entry).

The last equation can be rewritten as

$$\mathrm{val}\begin{pmatrix} P_pq_1U_1 & C_p-C_s+P_sq_1(U_1-\dfrac{C_p}{P_p}) \\ P_pqU_1 & C_p-C_s+(1-\alpha_s)P_sq_1(U_1-\dfrac{C_p}{P_p}) \\ P_pq_2U_1 & C_p-C_s \end{pmatrix} = C_p \qquad \text{(IV.42)}$$

Some useful conclusions about the optimal strategies can be reached without really solving Eq. (IV.42). One such conclusion refers to the condition which the parameters of the problem should satisfy in order for the game to have a pure optimal pair of strategies (which then should be AP for the attacker, and P_1-S_1 for the defender, as is obvious). It is conceivable that if the cost of an anti-secondary missile is "very high", then the attacker will tend to give up using it, and

thus will adopt a pure anti-primary strategy. Just how much is a "very high" cost of an anti-secondary missile in order for that to be true, can be discovered quite simply. We notice that since $q_1 < q < q_2$, the three entries in the first column of the matrix shown in Eq. (IV.42) form an increasing triplet (from top to bottom). Also, it can very easily be verified that the three entries in the second column form a decreasing triplet. Thus, a necessary and sufficient condition for the first column to dominate the second is that the (1,1) entry will be greater than the (1,2) entry, that is:

$$P_p q_1 U_1 > C_p - C_s + P_s q_1 (U_1 - \frac{C_p}{P_p}) \ . \qquad \text{(IV.43)}$$

(Notice that the matrix in Eq. (IV.42) is a transformation of the original matrix of the game. This transformation involves one alternation of sign. Therefore domination of one column of (IV.42) on the other means that the entries of the dominating column are greater than the corresponding entries of the dominated column. In the original matrix, domination between columns means just the opposite relation, since the entries there express costs, which the attacker, who selects the column, wishes to minimize).

Notice now the relation:

$$U_1 \leq \frac{C_p}{P_p q_1} = U_1^{AP}$$

To show the validity of this we need only to observe that the quantity U_1^{AP} is the expected cost of destruction when the

attacker pursues a pure AP strategy. The optimal cost of destruction can only be less than or equal to that quantity. Satisfaction of (IV.43) by U_1^{AP} is clearly a necessary condition for the optimal attacker's policy to be a pure AP one. Therefore the condition

$$P_p q_1 U_1^{AP} \leq C_p - C_s + P_s q_1 \left(U_1^{AP} - \frac{C_p}{P_p}\right)$$

which is equivalent to

$$C_s \leq \frac{P_s(1-q_1)}{P_p} \cdot C_p \qquad \text{(IV.44)}$$

is a sufficient condition for the optimal attacker's strategy to be a mixture of AP and AS decisions. Inequality (IV.44) provides a very simple and practical criterion with which one may check whether the use of AS-missiles is justified (notice, however, that (IV.44) is not a necessary condition for the beneficiality of AS-missiles).

We proceed by showing how to calculate the value U_1 for the case in which inequality (IV.44) holds true. A careful analysis of the matrix of the game reveals that there is no optimal (randomized) defensive policy which randomizes over the first and third rows only (see Eq. (IV.42)). This is the same property that has been proven for the other ASAPA game models (Sections E-F). It means that an optimal defense policy cannot consist of P_1-S_2 and P_2-S_2 response programs. The interpretation of this property is that when $\alpha_p \neq 1-\alpha_s$, the

two types of missiles are distinguishable, and this distinguishability must be exploited in the optimal defensive behavior. A mixed policy which contains only the P_1-S_1 and P_2-S_2 response programs is a policy which does not exploit that distinguishability, since both P_1-S_1 and P_2-S_2 programs do not refer to classification of missile type.

We omit a proof of the above property in this case. The method of proof is similar to that presented in the parallel theorems presented in Sections E and F. The only differences are algebraic.

As a result of this conclusion we can solve Eq. (IV.42) by the following method. We define U_1' as the solution of the equation:

$$C_p = \mathrm{val}\begin{pmatrix} P_p q_1 U_1 & C_p - C_s + P_s q_1 (U_1 - \frac{C_p}{P_p}) \\ P_p q U_1 & C_p - C_s + (1-\alpha_s) P_s q_1 (U_1 - \frac{C_p}{P_p}) \end{pmatrix} \qquad \text{(IV.45)}$$

and let U_1'' be the solution of:

$$C_p = \mathrm{val}\begin{pmatrix} P_p q U_1 & C_p - C_s + (1-\alpha_s) P_s q_1 (U_1 - \frac{C_p}{P_p}) \\ P_p q_2 U_1 & C_p - C_s \end{pmatrix} \qquad \text{(IV.46)}$$

By expressing explicitly the value of each of the matrices appearing in Eqs. (IV.45) and (IV.46), we get two quadratic equations in the variable U_1. Taking the maximum of the two solutions we arrive at the value of the game, which is:

$$U_1 = \mathrm{Max}[U_1', U_1''] .$$

PART TWO: OPTIMAL ALLOCATION PROBLEMS INVOLVING REAL MISSILES AND DECOYS

V. OPTIMAL DEPLOYMENT OF DECOYS--GENERAL INTRODUCTION

A. BACKGROUND

In Part I of this dissertation we dealt with missile allocation processes, in which the missiles of the anti-secondary type served as penetration aids. That is, they were tools used to facilitate the penetration of the main weapons (i.e., the anti-primary missiles). In Part II, a different concept of penetration support, namely the decoy, is analyzed. To recognize the coherence of these two parts, one should bear in mind that anti-secondary missiles and decoys represent nothing more than just two different technical approaches to the problem of improving penetration capability of the anti-primary missiles.

The principle of a decoy is simple. The decoy has the same physical signature as has the real missile, and thus it produces similar signals on the detection devices. The defener is incapable of distinguishing between a real missile and a decoy, and may be inclined to treat all detected missiles as if they were real ones. This fact works to the benefit of the attacker. In principle, there are two different basic effects through which this benefit may actually be gained:

(1) The Exhaustion Effect. This effect is significant when the defender has a limited number of defensive

missiles with which to counter the attacking missiles. The decoy is a device used to force, or to tempt, the defender to spend some or all of his missiles before the stockpile of real (offensive) missiles is exhausted. The attacker thus hopes that some missiles will be left for the attack after the defender is no longer capable of intercepting them.

(2) Saturation Effect. This is the effect induced by launching simultaneously a number of missiles (more than the number which the defender can simultaneously handle). The simplest case, which is quite realistic in many situations, is that in which the defender can engage only one missile at a time.

Chapter VI is devoted to modeling the exhaustion effects of decoys in a scenario similar to that which was considered in Chapters II through IV. In Chapter VII we present and solve some of the more important models of saturation effect.

Very naturally we expect decoys to cost less than a real missile, otherwise there is no point whatsoever in using them. In fact, decoys are devices which in most cases are simply designed to achieve tactical goals at a lower cost. It is true that whenever a real missile is replaced by a decoy in an actual combat process, the operational effectiveness is somewhat reduced. It is usually expected, on the other hand, that that loss of effectiveness will be much less significant

than the savings earned by that replacement, so that overall, decoys will prove cost-effective.*

It will be shown that the problems in which the exhaustion effect is involved very naturally call for a stochastic-game formulation and techniques. The problems in which the saturation effect is the main concern can be formulated as deterministic dynamic programming problems, as we show in Chapter VII.

B. O.R. LITERATURE ON DECOYS

It is a notable fact that the subject of optimal deployment of decoys in missile warfare is almost completely ignored in the open literature. Although there are aspects of decoy deployment, mainly technical ones, which are normally regarded as classified, and thus are restricted to appear in classified publications only , it is still hardly understandable that so little has been done openly. It seems that much work remains to be done that is not restricted to the classified literature. This is especially true for problems in which basic operational concepts of decoys deployment and methodologies of their effectiveness evaluation are involved.

We found only two papers in the O.R. literature in which the effect of decoys is the main theme. Both papers deal with problems taken from the area of anti-ballistic-missiles (ABM) vs. reentry vehicles (RV's), which is a very natural area for

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*For the same "economic" reasons, decoys are also widely used in other areas of warfare, as Electronic Warfare, Mine Warefare, passive Defense, etc.

the decoy concept to be applied. We give here a brief summary of these two papers and later on point out the differences between our approach to the analysis of decoys and the approach used in these papers.

1. Gorfinkel's Model

Gorfinkel [19] presents the following model: a "cloud": or reentry vehicles (RV's) approach the atmosphere aimed at an area defended by a given number of interceptors (that is, anti-ballistic missiles). Only one RV has a real warhead and all the others are decoys. The defense makes a measurement on each vehicle; this measurement is a random variable, the distribution of which depends on the type of missile which is actually being used (real|decoy). On the basis of this measurement the defense decides how many of its ABM's to divert to each vehicle. An ABM diverted to any vehicle kills it with probability P_k. It is assumed that the vehicles are sufficiently spread out in time so that a decision must be made on each one while those remaining are almost totally hidden. The opposite situation--when all objects are in view at once--is also discussed in the paper. As soon as a vehicle is considered and the 'proper' number of ABM's sent toward it, it is no longer counted as part of the cloud and the ABM's directed at it are no longer considered as part of the defensive arsenal.

A 'state' in the above process is characterized by the number i of RV's left to be considered, and the number j of ABM's still available to the defender. A solution of Gorfinkel's

problem tells how many ABM's to divert to an RV in state i-j as a function of the observed measurements on the RV. The criterion for developing the solution is to minimize the probability that the real warhead will penetrate.

Gorfinkel arrives at an analytic solution of the above problem, but the validity of his solution depends on some restrictive and rather artificial mathematical assumptions. These assumptions are:

(1) The existence of two distributions, $f_W(x)$ and $f_D(y)$, both <u>known</u> to the defender, of random variables X and Y which represent the intensity of signals coming from the real warhead and the decoy, respectively.

(2) The monotonicity of the likelihood ratio, which means that the ratio $f_W(x)/f_D(x)$ is a monotone increasing function of X.

In a subsequent unpublished paper, Gorfinkel [20] has postulated a more general model, by which the defense is assumed to know that exactly W real weapons are present among n offensive objects of the attack. He determined the optimum allocation strategies for two different criteria: minimizing the probability that at least one of the real weapons will penetrate, and minimizing the expected number of real weapons which penetrate.

2. Layno's Model

Layno [21] presents a different ABM allocation model. In his model the defense is facing a "cloud" of RV's, R of

them are real and D are decoys. The defense has I interceptors (IBM's). Each object in the cloud is detected and classified as either a real weapon or a decoy. Layno assumes also two types of error: Type 1 (mistaking a decoy for a real) and Type 2 (mistaking a real for a decoy). The probabilities P_1 and P_2 of the two types of errors are given. There is also a known probability of killing an object by a single interceptor. Layno's problem is to find numbers x (the number of ABM's assigned to each object classified as real) and y (the number assigned to each object classified as decoy) so as to minimize the expected number of real objects penetrating the defense. The variables x and y are constrained, of course, by the total number of interceptors available.

Layno solves the above problem using simple optimization techniques. His emphasis is on the relation between the ability to distinguish real missiles from decoys (which is reflected by P_1 and P_2) and the optimal expected number of penetrators. Layno compares the case in which discrimination capability does exist with the no-discrimination-capability case. His main conclusion is that even a modest discrimination capability significantly improves the payoff (i.e., the number of penetrating real missiles) over the non-discrimination case.

C. MODELS PRESENTED IN THIS THESIS

The whole subject of decoys is viewed, in this thesis, within a different context, i.e., the context of optimal

deployment of offensive tactical missiles. Our approach is offense oriented, as opposed to the defense oriented problems treated in the literature. The two papers mentioned above attempt to find the best way for the defender to cope with the presence of decoys in a "cloud" of threatening objects, whereas the main concern in this thesis is the explanation of the best deploying policies by which an attacker can deliberately exploit the fact that a decoy is hard to distinguish from a real weapon.

A general description of the modeling effort which has been carried out in this thesis is given in the following scheme:

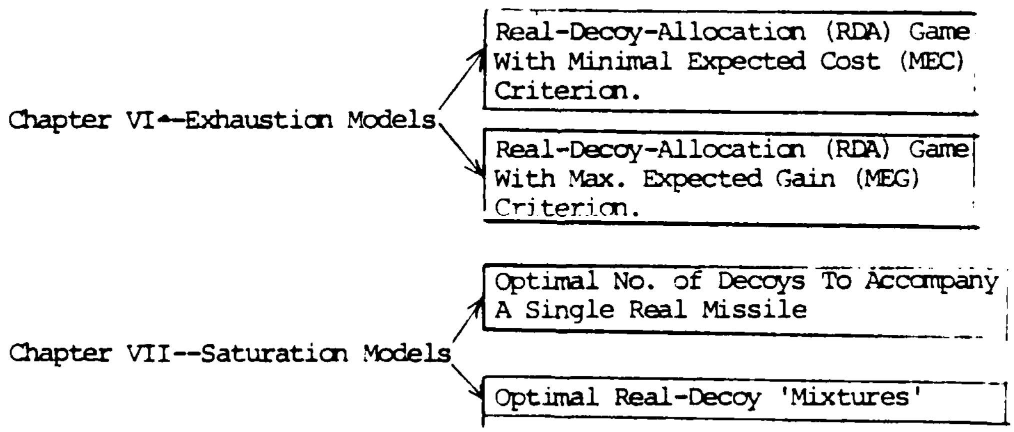

Fig. V.1: List of Decoy Models in the Thesis

As we see in the scheme, Chapter VI is devoted to modeling the exhaustion mechanism through which decoys might help the attacker to pursue a more cost-effective tactic. Two Real-Decoy-Allocation (RDA) games are formulated. In each of them

the attacker chooses, at every stage, to launch either a real, anti-primary missile or to use a decoy instead. The defender, being aware of the possibility of decoy deployment by the attacker, can choose either to fire a defensive missile (or salvo, whichever is technically relevant to him), or to hold his fire (anticipating that the next offensive object to be launched is a decoy). Decoys are assumed to be indistinguishable from real missiles. The defender is restricted by the number of missiles available to him. He cannot use more than one defensive missile (or salvo) against each offensive missile. In Chapter VI this RDA game is analyzed and solved for two different payoffs:

(1) The payoff is the cost of destroying the primary target. We assume here that the game is allowed to go on until the attacker achieves a hit on the primary target. The cost is the sum of all costs of the weapons (either real missiles or decoys) which are used. The attacker is seeking a policy of deployment which minimizes the expectation of this cost.

(2) The payoff is a linear function combining the military "worth" of the primary target and the cost of the weapons the attacker uses in his attempt to kill the the primary target. In this case we assume that the attacker is limited in the number of real missiles (but not of decoys) he can use. Therefore the success in destructing the primary target is not certain. Thus,

the probability of killing the target is entered into the payoff function along with the cost of the weapons.

Chapter VII is devoted, as seen in the scheme given above (Fig. V.1) to modeling the saturation effect. The fundamental problem we raise is what should be the optimal "mixture" of real missiles and decoys in a "wave" of attacking objects which are to be launched simultaneously, so as to minimize the expected cost of destroying the primary target (assuming that attacking "waves" are to be launched repeatedly until the primary target is finally hit)? As a preliminary problem, we analyze a model in which it is assumed that the attacker is restricted to launch only one real missile, which he can accompany by any desired number of decoys. This restricting assumption may very well be realistic in case the real missile is a very costly and scarce weapon, and the decoys for it are very inexpensive and available in large quantities. The question in that preliminary problem is to find the optimal number of decoys to be launched simultaneously with a real missile. Two effects, associated with launching more decoys, are competing here with each other:

(1) More decoys means higher cost.

(2) More decoys means high probability of survival of the real missile.

In solving this problem we distinguish between two cases, corresponding to two different schemes of operations for the defense; the first case is one in which all secondary targets are assumed to operate independently. In the second case,

coordination is assumed to exist among the defense units, so that no single offensive object (either real missile or a decoy) is likely to be engaged by more than one secondary target.

The saturation models of Chapter VII are treated by methods of dynamic programming. The dynamic programming arguments give rise to various functional equations which are then solved in detail. Operational interpreation of the results is also included.

VI. OPTIMAL DEPLOYMENT OF DECOYS--MODELING THE EXHAUSTION EFFECT

A. THE REAL-DECOY ALLOCATION (RDA) GAME--MINIMUM EXPECTED COST (MEC) CRITERION

1. Formulation

Suppose that a single primary target is defended by a defense system (which may consist of one SAM battery or more), and suppose that the defense is limited to a given number of intercepting missiles (or salvos) that may be launched throughout the process. We make the following assumptions about the process:

(a) At each stage, the attacker launches one missile, which is either a real one or a decoy, according to the attacker's choice.

(b) The defender can react either by firing a defensive missile (or salvo) on the offensive object, or by holding his fire. For technical reasons, he cannot launch more than one defensive missile (salvo) on a single offensive object.

(c) The decoy is indistinguishable from a real missile.*

(d) There is perfect information to both sides (as is assumed everywhere in this thesis).

We use the following notation:

C_R - cost of a real missile

C_D - cost of a decoy ($C_D < C_R$)

N - Number of intercepting missiles (salvos) available to the defender

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*This assumption is implied by the specific type of decoy that we have in mind. A decoy is thought of as an object identical with a real one in its physical signatures, but lacking the sophisticated guidance system, warhead and some other elements that the real missile does have.

q - (As before)--Probability that a real missile survives an interception attempt made by the defender.

P - Probability that the target is killed by a real missile, given that it survives an interception attempt.

It is assumed that the game is allowed to continue until the primary target is killed. The payoff of the game is the total cost paid for achieving the destruction of the primary target. The attacker wishes to minimize the maximum expected cost of killing the target. Let us denote by Γ^N the game element played when the defender has exactly N intercepting missiles (salvos) left. We can write Γ^N in the following matrix form:

Defender's choice \ Attacker's choice	Real	Decoy
Fire	$C_R + (1-Pq)\Gamma^{N-1}$	$C_D + \Gamma^{N-1}$
Hold Fire	$C_R + (1-P)\Gamma^N$	$C_D + \Gamma^N$

The expressions in the four entries of the matrix are readily verified. We explain two entries as examples:

(a) If the attacker launches a real missile, and the defender fires at it, an immediate cost C_R is incurred by the attacker, and in addition, the game Γ^{N-1} is played at the next stage with probability 1-Pq, which is the probability of missing the primary target with the real missile.

(b) If a decoy is launched, and the defender holds fire, a cost C_D of the decoy is incurred, and in addition, the stochastic game stays (with certainty) at state Γ^N to the next stage.

2. General Solution

We replace Γ^N by its value V_N, and get the equation:

$$V_N = \text{val}\begin{pmatrix} C_R + (1-Pq)V_{N-1} & C_D + V_{N-1} \\ C_R + (1-P)V_N & C_D + V_N \end{pmatrix} \qquad \text{(VI.1)}$$

Using formula (1) of the Appendix we can explicitly write the value above as

$$V_N = \frac{[C_R - C_D(1-P) + P(1-q)V_{N-1}]V_N - [C_R - C_D(1-Pq)]V_{N-1}}{P(V_N - q\cdot V_{N-1})} \qquad \text{(VI.2)}$$

or

$$V_N^2 - \left(\frac{C_R - C_D(1-P) + P\cdot V_{N-1}}{P}\right)V_N + \frac{C_R - C_D(1-Pq)}{P}\cdot V_{N-1} = 0 \quad . \qquad \text{(VI.3)}$$

It is convenient to work out this problem, using dimensional analysis. To accomplish that, we define:

$$\bar{V}_N = \frac{V_N}{C_R/Pq}\,, \qquad r_c = \frac{C_D}{C_R}\,.$$

Notice that $\bar{V}_N$ is the cost of destruction measured in terms of the cost of destruction that would be incurred if the attacker used real missiles only. Thus $\bar{V}_N$ is a very natural dimensionless quantity, with which we can evaluate the contribution of decoys to the cost-effectiveness ratio. Clearly, $\bar{V}_N$ is always less than (or equal to) <u>one</u>. The parameter r_c is the ratio of costs of a decoy and a real missile, and its value always lies between zero and one.

We define also:

$$B = q - qr_c(1-P) \,, \qquad C = q-qr_c(1-Pq)$$

Eq. (VI.3) thus becomes:

$$\bar{V}_N^2 - (B + \bar{V}_{N-1}) \cdot \bar{V}_N + C \cdot \bar{V}_{N-1} = 0 \,. \tag{VI.4}$$

This equation is readily solved for $\bar{V}_N$:

$$\bar{V}_N = \frac{B + \bar{V}_{N-1} \pm \sqrt{(B+\bar{V}_{N-1})^2 - 4C\bar{V}_{N-1}}}{2} \,. \tag{VI.5}$$

As Eq. (VI.5) shows, two solutions exist to Eq. (VI.1). Only one of them is significant (The possibility of non-unique solution to the value-equation always exists when there are entries with zero probability of stopping, as we have in this stochastic game. Shapley [5] shows uniqueness only under the assumption of non-zero probability of stopping. See Section IV.B.)

We shall prove now that the correct value of the game Γ^N corresponds to the plus sign in Eq. (VI.5) above. Observe first that for all N we have:

$$\bar{V}_N > \bar{V}_{N-1} \,. \tag{VI.6}$$

This relation is obvious since the more intercepting missiles (salvos) the defender has, the higher is the cost which the attacker is expected to pay for killing the primary target. Now notice that, since $q < 1$, we have:

$$B = q - qr_c(1-P) > q - qr_c(1-Pq) = C$$

Notice also that since $V_0 = C_R/P$, we have $\bar{V}_0 = q$, and hence $\bar{V}_N > \bar{V}_0 = q > B$. Therefore, if the minus sign were the correct one in Eq. (VI.5), we would have:

$$\bar{V}_N = \frac{B + \bar{V}_{N-1} - \sqrt{(B+\bar{V}_{N-1})^2 - 4C\bar{V}_{N-1}}}{2} < \frac{B+\bar{V}_{N-1} - \sqrt{(B+\bar{V}_{N-1})^2 - 4B\bar{V}_{N-1}}}{2}$$

$$= \frac{1}{2}\left(B + \bar{V}_{N-1} - \sqrt{(\bar{V}_{N-1} - B)^2}\right) = B < q = \bar{V}_0$$

The inequality $\bar{V}_N < \bar{V}_0$, which we got here, contradicts the obvious fact expressed in Eq. (VI.6) above. Thus, we have proven that if we assume that the minus sign in Eq. (VI.5) is the correct one, we are led to contradict an obvious property of the process (i.e., that $\bar{V}_N > \bar{V}_{N-1}$). Therefore, the plus sign should be the significant one:

$$\bar{V}_N = \frac{B + \bar{V}_{N-1} + \sqrt{(B+\bar{V}_{N-1})^2 - 4C\bar{V}_{N-1}}}{2} \qquad \text{(VI.7)}$$

It should be noticed that Eq. (VI.7) is meaningful only if the matrix of the game Γ^N doesn't have a saddle point. In what follows we derive the conditions for that matrix to have a saddle point. We write again the matrix of Γ^N as

$$\begin{pmatrix} C_R + (1-Pq)V_{N-1} & C_D + V_{N-1} \\ C_R + (1-P)V_N & C_D + V_N \end{pmatrix}$$

For convenience, we denote by e_{ij} (i,j = 1,2) the (i,j) entry of the above matrix. We notice immediately that

$$e_{22} > e_{12} \qquad \text{(VI.8a)}$$

(since $V_N > V_{N-1}$) and:

$$e_{21} < e_{22} \qquad \text{(VI.8b)}$$

To show this last inequality, assume the contrary for the moment, i.e., assume that $e_{21} \geq e_{22}$, i.e.,

$$C_R + (1-P)V_N \geq C_D + V_N$$

or

$$V_N \leq \frac{C_R - C_D}{P} < \frac{C_R}{P} = V_0$$

which contradicts the obvious fact that $V_N > V_0$, hence Eq. (VI.8b) is correct. Now, inequalities (VI.8a) and (VI.8b) show that only e_{11} and e_{21} can be saddle points. But, we can exclude the possibility that entry e_{21} is a saddle point, for if it were, the value V_N would be equal to that entry (see Eq. (VI.1)), so that we would have:

$$V_N = C_R + (1-P)V_N \qquad V_N = \frac{C_R}{P} = V_0,$$

which contradicts the fact that $V_N > V_0$. Thus, it is impossible that $e_{21} > e_{11}$ (otherwise, together with Eq. (VI.8b) it would have implied that e_{21} is a saddle point). The only entry

which may be a saddle point is therefore, e_{11}. The necessary and sufficient condition for a saddle point to exist is:

$$e_{11} < e_{12},$$

or

$$C_R + (1-Pq)V_{N-1} < C_D + V_{N-1}$$

which is equivalent to:

$$V_{N-1} > \frac{C_R - C_D}{Pq} \Rightarrow \bar{V}_{N-1} > 1-r_c \; .$$

It should be noted that the fact that only e_{11} can be a saddle point could be quite easily argued by discerning the nature of the process. It is clear that the only pair of pure defense-offense strategies which can exist as an equilibrium pair, is the "Fire"-"Real" pair. All other three pairs are "unstable" in the sense of game theory. For instance, the pair "Fire"-"Decoy" is unstable because the defender would incline to alter his action and avoid firing if he knew that his opponent was using decoys only. Similarly, for the "Hold Fire"-"Decoy" pair of strategies, the attacker would prefer using a real missile if he knew that the defender was holding his fire.

We define now the value N* by

$$N^* = \mathrm{Min}\{N: \; \bar{V}_N > 1-r_c\} \qquad \text{(VI.10)}$$

The solution to the problem can now be summarized as follows: For $N \leq N^*$ the sequence of values is given by the recursive equation (VI.7), starting from $\bar{V}_0 = q$. The optimal strategies are randomized, and can be calculated directly, using formulae of the Appendix. We denote by $\pi^{of}(N)$ the probability with which the attacker has to choose to launch a real missile if he adopts the optimal strategy, and by $\pi^{def}(N)$ the probability with which the defender has to choose to fire. Formulae 2 and 3 in the Appendix give:

$$\pi^{of}(N) = \frac{V_N - V_{N-1}}{P(V_N - qV_{N-1})} = \frac{\bar{V}_N - \bar{V}_{N-1}}{P(\bar{V}_N - q\bar{V}_{N-1})} \qquad \text{(VI.11a)}$$

$$\pi^{def}(N) = \frac{PV_N + C_D - C_R}{P(V_N - qV_{N-1})} = \frac{\bar{V}_N + qr_C - q}{\bar{V}_N - q\bar{V}_{N-1}} . \qquad \text{(VI.11b)}$$

For $N > N^*$, the optimal strategies are pure: The attacker uses a real missile and the defender fires at it--both taking their decisions with probability one. Notice that N^* is a function of the parameters P, q, r_C.

At that point it seems appropriate to provide some intuitive interpretations of the optimal strategy established above. The main point which deserves elaboration is the existence of the number N^*. The optimal policy (viewed as a function of N) dictates that if the defender has more than N^* missiles to counter the attacking missiles, the attacker should use real missiles only. This may seem somewhat contradictory to the rather simplistic intuitive notion that the more defensive

missiles there are, the more desirable it becomes to use decoys to exhaust them. This is not so in the model we have discussed here. The reason for that is, that if the defender is equipped with too many missiles, the option of exhausting him with decoys (and using real missiles later) becomes more expensive than the option of penetrating through the defense with real missiles (although this means accepting a lower probability of success). It can be argued that when the defender has many missiles, the marginal gain expected from using a decoy (which at most would reduce the number of missiles by one) is very insignificant; the defender would still have many missiles which would require too many decoys to exhaust. Using a real missile, instead, would still provide the attacker with some probability of success (although lower than the probability he could enjoy if he exhausted the defender), and of course, would save (with some probability) the cost of all those decoys which are required for exhaustion.

The value of N^* is the precise indicator of what should be considered as "many" (defensive missiles) in the above argument. We turn now to analyze, by intuitive arguments, the dependence of N^* on the parameter r_c. First, if $r_c = 1$, or if r_c is very close to 1, it means that a decoy is as expensive (or almost as expensive) as a real missile. Therefore, the attacker cannot have any motivation to use it in place of a real missile. The formal statement of this assertion is that $r_c \simeq 1$ implies $N^* = 0$. Defininition (VI.10) of N^* reveals that this is indeed the case. To be more precise, notice that the

definition of N^* (VI.10) implies that a necessary and sufficient condition for N^* to be equal to zero is:

$$\bar{V}_0 = q > 1 - r_c$$

or

$$r_c > 1-q \tag{VI.12}$$

On the other hand, if $r_c = 0$ (or r_c is very close to zero) it is conceivable that the attacker would prefer to exhaust the defender, even if this requires spending many decoys, as is the case when the defender has many missiles. This intuitive observation is formally expressed by N^* being very large. In the most extreme case (that is, the case $r_c = 0$), N^* should be infinite. The practical meaning of this statement is that no matter how many intercepting missiles the defender has, the attacker will use decoys only until the defender is exhausted. This is obvious because he pays nothing for decoys. The equation defining N^* indeed approves this assertion: since when $r_c = 0$, definition (VI.10) reduces to:

$$N^* = \mathrm{Min}\{N;\ \bar{V}_N > 1\},$$

and as was shown before, there is no value of N such that $\bar{V}_N > 1$. Thus, $N^* = \infty$.

The value of N^* is thus seen to increase from 0 to ∞, as r_c decreases from 1 to 0.

Inequality (VI.12) given above provides a very useful and convenient criterion for the attacker to decide on whether decoys are at all worth acquiring and deploying. This inequality simply says that if a decoy is more expensive than 1-q times the cost of a real missile, which it is designed to imitate, then it is not worth deploying. It also indicates the fact which is quite expected, namely, that as the probability of survival gets higher, the limit on acceptable decoy costs gets lower (or the need for decoys is lessened).

We now turn to calculating the value of the game Γ^N for values of N greater than N*. It is given by:

$$V_N = C_R + (1-Pq)V_{N-1} \quad . \qquad \text{(VI.13)}$$

This equation is a simple linear difference equation. Assuming that V_{N^*} is already known (after performing the recursive series of calculations using Eq. (VI.7), from N = 1 to N = N*), we find for N greater than N*:

$$V_N = C_R+(1-Pq)(C_R+(1-Pq)\cdot V_{N-2}) = C_R+(1-Pq)C_R+(1-Pq)^2 V_{N-2}$$

$$= C_R+(1-Pq)C_R+(1-Pq)^2[C_R+(1-Pq)V_{N-3}]$$

$$= C_R+(1-Pq)C_R+(1-Pq)^2 C_R+(1-Pq)^3 V_{N-3}$$

$$= \ldots = \sum_{i=1}^{N-N^*} (1-Pq)^{i-1} C_R + (1-Pq)^{N-N^*}\cdot V_{N^*}$$

$$= C_R\cdot\frac{1-(1-Pq)^{N-N^*}}{Pq} + (1-Pq)^{N-N^*}\cdot V_{N^*}$$

or

$$V_N = \frac{C_R}{Pq} - (1-Pq)^{N-N^*}(\frac{C_R}{Pq} - V_{N^*}) ,$$

which is equivalent to

$$\overline{V}_N = 1 - (1-Pq)^{N-N^*}(1 - \overline{V}_{N^*}) . \qquad \text{(VI.14)}$$

3. Numerical Examples

The numerical examples presented in this section are intended to give graphical depiction of the following relations:

(a) The value $\overline{V}_N$ as a function of r_c (decoy-to-real cost ratio), for various values of q (probability of survival) and N (no. of defensive missiles). These functions are shown in Figs. VI.1 through VI.4. Each figure corresponds to a different value of q (q = 0.2, 0.4, 0.5, 0.6) and shows four curves, corresponding to N = 1, 2, 3, 4.

(b) The probability that the attacker selects a real missile for launch (π^{of}), as a function of r_c. This is done on Figs. VI.5 (for q = 0.2) and VI.6 (for q = 0.6). Here again, each figure shows four curves, corresponding to N = 1, 2, 3, 4.

(c) The probability that the defender chooses to fire a defensive missile (π^{def}), as a function of r_c. This is given on Figs. VI.7 and VI.8, for the same combinations of values of q and N as in (b) above.

All the numerical examples shown here were calculated assuming P = 0.5. The method of calculation we used follows exactly the logic presented in Section A.2, and is summarized below. For each pair of values q, r_c (assuming $r_c < 1-q$) we do the following:

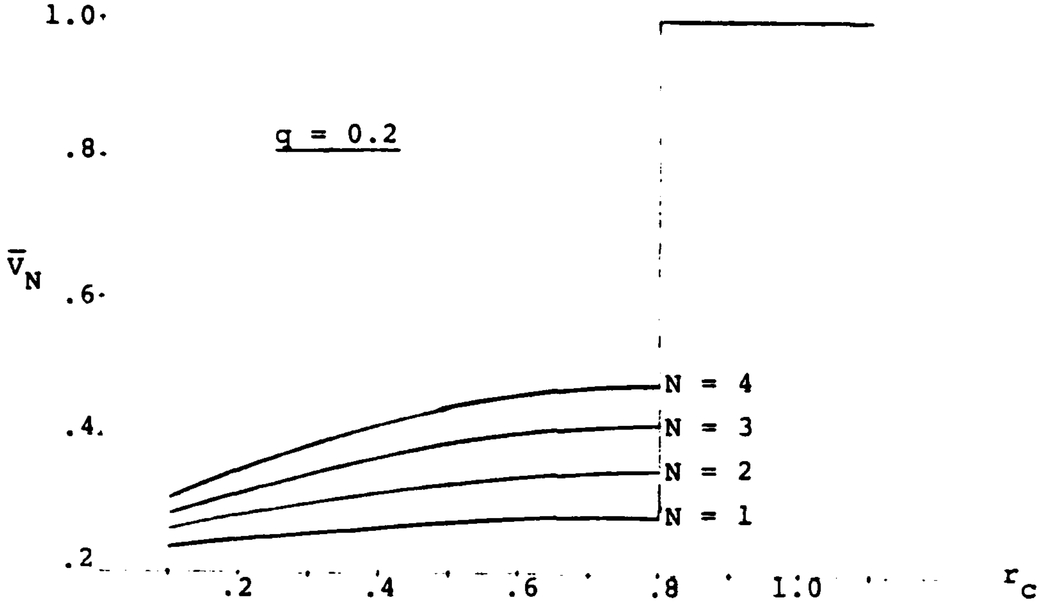


Figure VI.1: Normalized Value of the Real-Decoy-Allocation (RDA) Game as Function of Decoy-to-Real Cost Ratio for q = 0.2

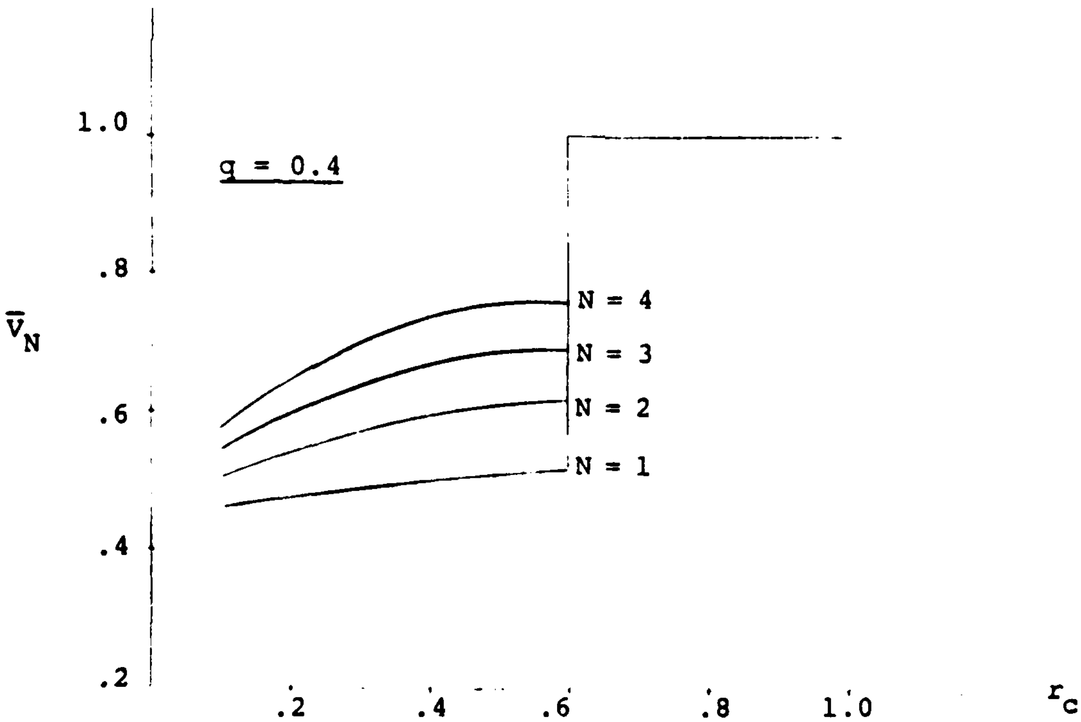


Figure VI.2: Normalized Value of the Real-Decoy-Allocation (RDA) Game as Function of Decoy-to-Real Cost Ratio, for q = 0.4

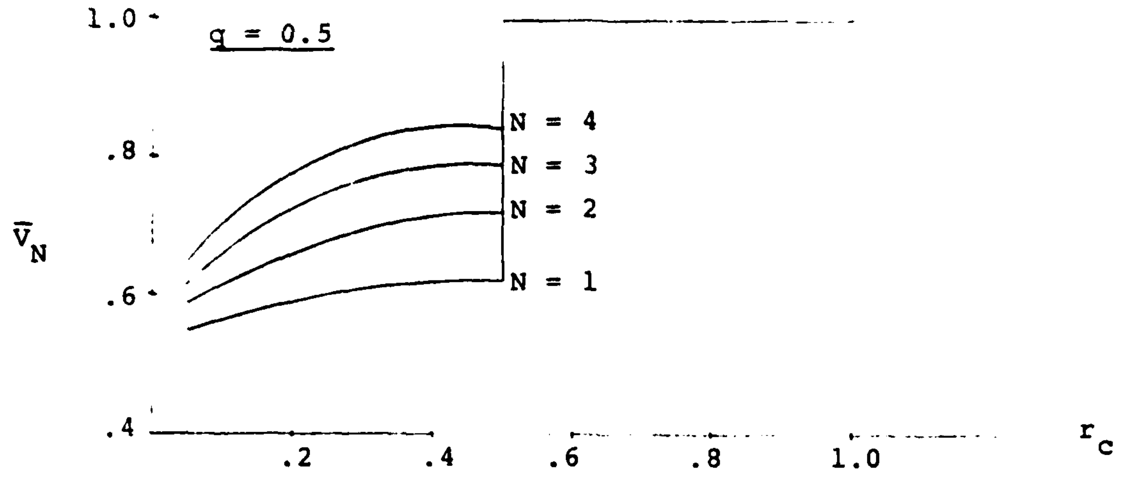


Figure VI.3: Normalized Value of the Real-Decoy-Allocation (RDA) Game as Function of Decoy-To-Real Cost Ratio, for q = 0.5

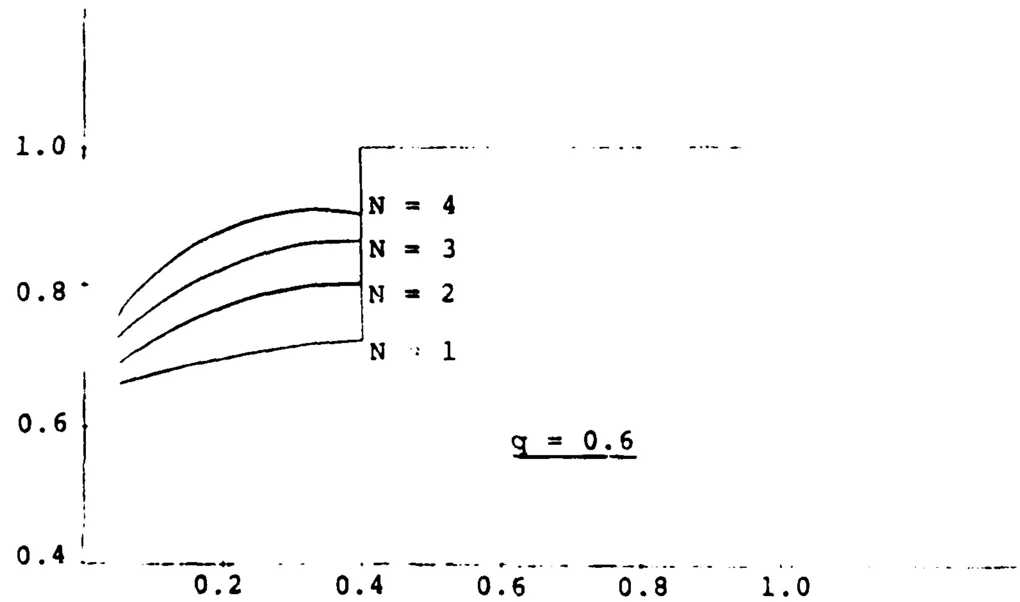


Figure VI.4: Normalized Value of the Real-Decoy-Allocation (RDA) Game as Function of Decoy-to-Real Cost Ratio for q = 0.6

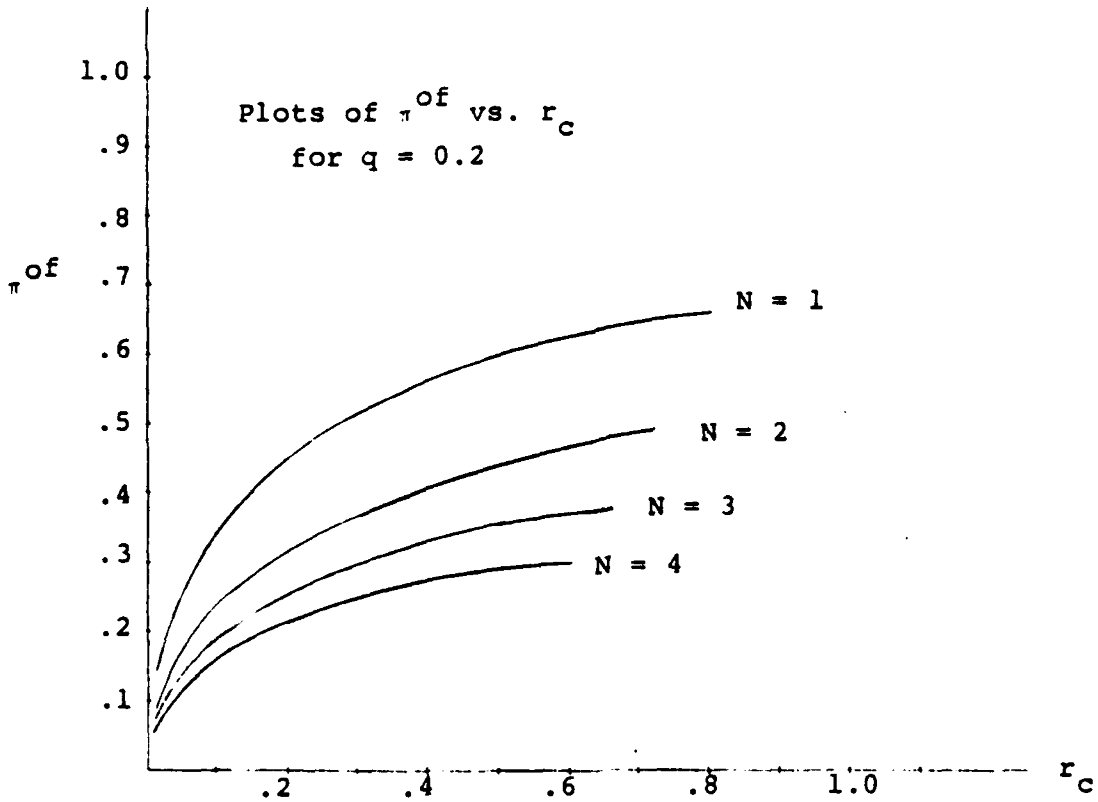


Figure VI.5: Optimal Prob. of Real Missile Launch for the Real-Decoy-Allocation (RDA) Game as Function of Decoy-To-Real Cost Ratio, for q = 0.2

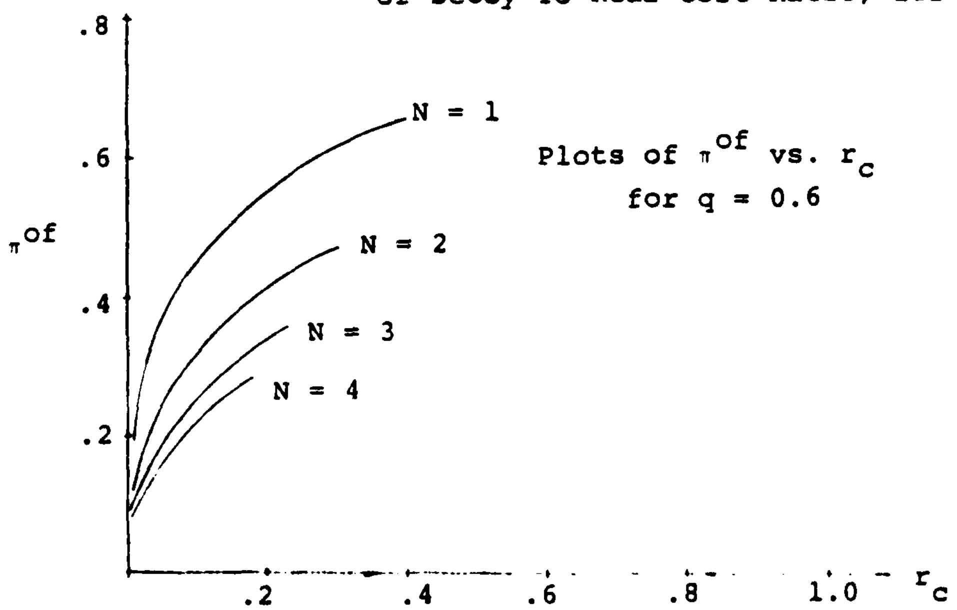


Figure VI.6: Optimal Prob. of Real Missile Launch for the Real-Decoy-Allocation (RDA) Game as Function of Decoy-To-Real Cost Ratio, q = 0.6

FIGURE VI.7: Optimal Prob. of Defensive Firing in the Real-Decoy-Allocation (RDA) Game, as Function of Decoy-To-Real Cost Ratio, for q = 0.2

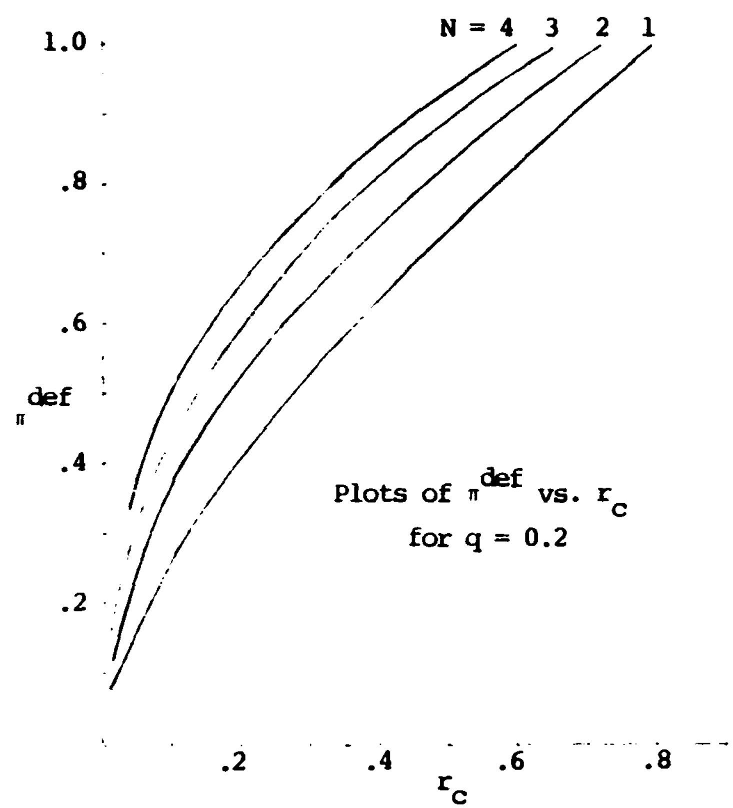
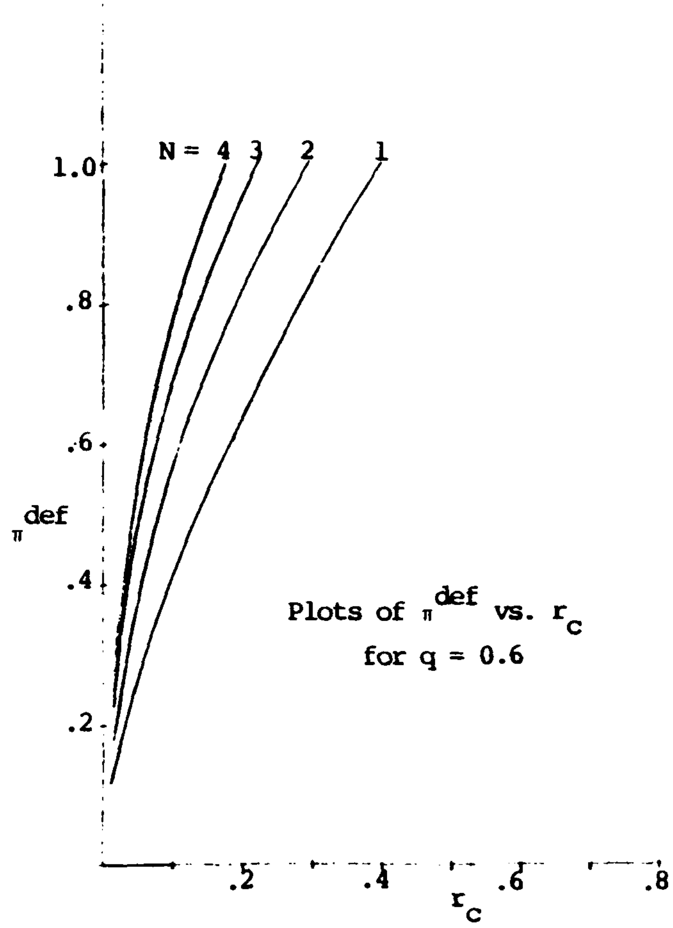


FIGURE VI.8: Optimal Prob. of Defensive Firing in the Real-Decoy-Allocation (RDA) Game, as Function of Decoy-To-Real Cost Ratio, for q = 0.6



(a) Calculate B and C (the parameters appearing in Eq. (VI.5)) from P (= 0.5), q, r_c.

(b) Set $\overline{V}_0 = q$, N = 0.

(c) Calculate $\overline{V}_N$ from $\overline{V}_{N-1}$ (and the parameters B, C) using Eq. (VI.5).

(d) Calculate $\pi^{of}(N)$ and $\pi^{def}(N)$ using eqs. (VI.11a) and (VI.11b).

(e) Check whether $\overline{V}_N > 1-r_c$ (see Eq. (VI.10)). If the answer is negative, set N = N+1 and go to step (c). If positive, set N* = N.

(f) For all values of N greater than N*, calculate $\overline{V}_N$ using Eq. (VI.13).

Notice that we refer to $\overline{V}_N$ as the most natural measure of the decoys effectiveness as exhausting devices. The value $\overline{V}_N$ gives the cost of destroying the target with decoys (deployed optimally) relative to what it would cost to do without them.

4. Analysis of Results

In Figs. VI.1 through VI.4 we note that the graphs of $\overline{V}_N$ jump discontinuously to $\overline{V}_N = 1$ on $r_c = 1-q$, for all values of q and N. This reflects the fact that if $r_c \geq 1-q$, the optimal deployment policy uses only real missiles (as proven in Section A.2). Notice also that for any pair of r_c and N, the contribution of decoys becomes more significant as q gets smaller, i.e., $\overline{V}_N$ is decreasing with q, for fixed r_c, and for all N.

It is also interesting to point out, that in practical terms, $\overline{V}_N$ proves to be not very sensitive to the cost of a decoy (that is, to r_c). If q = 0.2, for instance, and N = 1, $\overline{V}_1$ varies from .24 to .28 as r_c varies from 0.1 to 0.8. If

$q = 0.6$ (see Fig. VI.4), and $N = 1$, $\bar{V}_1$ varies from 0.67 to 0.72 as r_c varies from 0.1 to 0.4. For the same value of q, but with $N = 3$, $\bar{V}_3$ varies from 0.77 to 0.86 when r_c varies over the same region. The conclusion, which is somewhat surprising is that decoy cost does not have a very significant impact upon the overall cost of destruction. This conclusion is of course valid only so long as the value of r_c doesn't exceed 1-q. (For $r_c > 1-q$ we have $\bar{V}_N = 1$ for all N.)

In contrast with the above conclusion is the fact which may very easily be observed on Figs. VI.5-VI.8, that the optimal strategies of both the attacker and the defender are quite sensitive to r_c. In all those figures the curves are "cut" at exactly those values of r_c above which the optimal strategies are pure (i.e., the attacker should launch only real missiles and the defender should always fire).

From Figs. VI.5 and VI.6 we see that π^{of}, the probability of selecting a real missile to launch at the optimal strategy, is an increasing function of r_c (for each N). In other words, the probability of using decoys decreases as the cost of a decoy increases, which is also obvious intuitively. We observe also that as N increases, the probability of using real missiles decreases (hence, probability of using decoys increases).

If $N = 1$, we find that as r_c approaches 1-q (from below), the optimal defensive policy approaches the pure "fire" policy ($\pi^{def} = 1$), whereas the optimal policy for the attacker is

still a randomized policy. For example, if $q = 0.6$, and $N = 1$, we see from Fig. VI.8 that as $r_c \to 0.4$, π^{def} approaches 1, whereas from Fig. VI.6 we find that π^{of} approaches 0.66. Thus, for values of r_c very close to $1-q$ (but below it), a very peculiar feature of the exhaustion model is revealed. That feature is, that whereas the attacker's optimal policy still relies on the use of decoys to quite a significant extent, the defender's optimal policy dictates an almost pure 'fire' behavior. The point $r_c = 1-q$ is of course a point of discontinuity of π^{of} $(N = 1)$ (as well as of $\bar{V}_N$). For $r_c \geq 1-q$, we have π^{of} $(N = 1) = 1$, whereas for all points at which $r_c < 1-q$, the value of π^{of} is bounded by a value less than one.

An interesting observation is the following: Let u be defined as the value of π^{of} at $N = 1$, on points (r_c, q) for which $r_c = 1-q$. We show that u is independent of q (hence, also of r_c), and is a function of P only. (Indeed, we can see that on Figs. VI.5 and VI.6, the value of π^{of} on $N = 1$, $q = 0.2$, $r_c = 0.8$ (Fig VI.5) and the value of π^{of} on $N = 1$, $q = 0.6$, $r_c = 0.4$ (Fig. VI.6) are both equal to 0.66.)

To prove this asssertion, we make use of Eq. (VI.7). We substitute $N = 1$, $\bar{V}_0 = q$ and calculate B and C, which appear in that equation, directly from their definition:

$$B = q - qr_c(1-P), \qquad C = q - qr_c(1-Pq)$$

Putting $r_c = 1-q$ we get:

$$B = q(P + q - Pq), \qquad C = q^2(1 + P - Pq)$$

and thus we find from Eq. (VI.7):

$$\bar{V}_1 = \frac{q(P+q-Pq) + q + \sqrt{q^3(P+q-Pq+1)^2 - 4q^2(1+P-Pq)}}{2}$$

$$= q(1 + P - Pq).$$

Now substituting that last expression for $\bar{V}_1$ in the formula for π^{of} (Eq. (VI.11a)), we get:

$$u = \pi^{of}(N = 1, r_c = 1-q) = \frac{\bar{V}_1 - \bar{V}_0}{P(\bar{V}_1 - q\bar{V}_0)}$$

$$= \frac{q(1+P-Pq) - q}{P[q(1+P-Pq) - q^2]} = \frac{1}{1+P},$$

and so π^{of}, calculated for points (r_c, q) such that $r_c = 1-q$, is independent of q (or r_c). Another noteworthy property of the model described in this section is that

$$\pi^{of}(N) \xrightarrow[r_c \to 0]{} 0 \quad \text{for all } N, q,$$

and

$$\pi^{def}(N) \xrightarrow[r_c \to 0]{} 0 \quad \text{for all } N, q.$$

In other words, if the cost of a decoy is near zero, then the defensive optimal policy tends to be a pure "hold fire" policy and the attacker's optimal policy tends to be a pure decoys launch policy. At $r_c = 0$ the situation is--from the formal mathematical standpoint--one in which the value of the stochastic game exists but optimal policies do not. That this is so

can be intuitively argued, because a policy for which $\pi^{of}(N) = 0$, for some N, never leads to termination of the game. The attacker is supposed to spend an unlimited number of decoys, which would cost nothing, but would not lead to a target kill either. The values of all the games Γ^N $(N \geq 1)$ do exist however at $r_c = 0$, and are equal to q, as can be directly deduced from Eq. (VI.7).

It is true, here, although the proof will not be shown, that although optimal policies do not exist for $r_c = 0$, ε-optimal policies do, for every $\varepsilon > 0$ and every N. That is, for any $\varepsilon > 0$ there is a pair of strategies, such that the min-max cost is less than $\overline{V}_N + \varepsilon$, and greater than $\overline{V}_N - \varepsilon$.

In Table VI-1 we present values of N* for various combinations of r_c and q. The number N* is the maximal number of defensive missiles (or salvos) for which the attacker has to use decoys in his optimal strategy (see definition--Eq. (VI.10)). The table shows that as q and r_c become smaller, N* increases and reaches very high values (N* = 135 for $q = r_c = 0.1$). This fact, along with the monotone-increasing property of the function $1-\pi^{of}$, is the formal expression of the rather intuitive fact that as the survivability and decoy cost get smaller, the attacker would tend to make a heavier use of decoys.

Table VI-1

Values of N* for Various Combinations of r_c (Decoy-to-Real Cost Ratio) and q (Probability of Surviving an Engagement*

r_c \ q	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.1	135	55	30	18	11	7	4	2
0.2	60	24	13	8	5	3	2	-
0.3	35	14	8	4	3	1	-	-
0.4	22	9	5	3	1	-	-	-
0.5	15	6	3	1	-	-	-	-
0.6	10	3	2	-	-	-	-	-
0.7	6	2	-	-	-	-	-	-
0.8	3	-	-	-	-	-	-	-

*The probability of kill P is assumed equal to 0.5

B. THE REAL-DECOY ALLOCATION (RDA) GAME WITH LIMITED NUMBER OF REAL MISSILES

1. Formulation

We assume now, that the attacker has M real missiles, and in addition, he can spend decoys without any restriction on the number available to him. We can think of the decoys as very inexpensive dummy missiles so that it may reasonably be assumed that the attacker has relatively many of them.

The defender has N intercepting missiles (or salvos), and it is assumed as before that he can launch only one missile (salvo) at each stage. He may choose to fire or to hold fire. The process is terminated when the attacker either hits the target or has all his real missiles spent already, whichever comes first.

We use here all the parameters used in Section A; P, q, C_R, C_D. It is also assumed that the primary target has an operational value v. This means that if the target is killed, the attacker gains a value worth v units which must be the same units in which C_R and C_D are measured. In other words, the value v serves to make the worth of the primary target destruction comparable to the cost of the missiles. This enables us to combine both the missile cost and target value in one objective function, as we now show.

Let P_k be the probability that the primary target will eventually be killed. The P_k is, of course, a function of the policies of the defender and the attacker. Let C_T be the total cost, defined as the sum of all real missiles and decoys

consumed in the process. The cost is of course a random variable. Let Z be a random variable defined as:

$$Z = \begin{cases} 1 & \text{if the primary target is destroyed} \\ 0 & \text{if not.} \end{cases}$$

The function $vZ - C_T$ clearly describes the net gain of the attacker in any realization of the process. We take the expectation of this gain as the payoff of the game which is to be played, that is, the payoff (paid to the attacker) is:

$$E(vZ - C_T) = v \cdot P_k - E(C_T).$$

Let us denote by $\Gamma^{M,N}$ the game played when the attacker has M real missiles and the defender has N defensive missiles (salvos). We write

$$V_{M,N} = \mathrm{val}(\Gamma^{M,N})$$

The recursive relation which $V_{M,N}$ must satisfy is:

$$V_{M,N} = \mathrm{val}\begin{pmatrix} -C_R + Pq \cdot v + (1-Pq)V_{M-1,N-1} & -C_D + V_{M,N-1} \\ -C_R + P \cdot v + (1-P)V_{M-1,N} & -C_D + V_{M,N} \end{pmatrix} \qquad \text{(VI.15)}$$

where, as usual, the defender selects the row and the attacker selects the column; the first row corresponds to the defender's decision to fire, and the second corresponds to a decision to hold fire. The first column corresponds to the attacker's

decision to launch a real missile, and the second column corresponds to a decision to launch a decoy.

2. General Solution

We can assume, without loss of generality, that $v = 1$. We only have to remember that values of C_D, C_R are then measured in terms of the target "worth", that is, they are taken relative to the value of the target. The same is true for $V_{M,N}$. Thus we can take $v = 1$ from now on. We will always assume that $C_R < 1$ (since byassuming the opposite we reach an absurdity; the cost of a single weapon designed to kill some target cannot be greater than the value of killing that target).

Going back to Eq. (VI.15), notice that $V_{M,N}$ appears in both sides, so that by explicitly writing the right hand side, using Formula 1 of the Appendix, we will get a quadratic equation in $V_{M,N}$ with coefficients which are functions of $V_{M-1,N-1}$, $V_{M-1,N}$ and $V_{M,N-1}$. The solution of this quadratic equation gives a (non-linear) difference equation of the form

$$V_{M,N} = f(V_{M-1,N-1}, V_{M-1,N}, V_{M,N-1}) .$$

Thus, in order to calculate $V_{M,N}$, we have to have the values of $V_{M-1,N'}$ (for all N' up to N) and of $V_{M',N-1}$ (for all M' up to M) already calculated. The initial conditions can be derived as follows:

$$V_{0,N} = 0 \quad \text{for all } N .$$

This is so because if the attacker has no real missiles, no game actually has to be played. To calculate $V_{M,0}$, we first note that the optimal attacker's strategy in the case N = 0 (no defense exists) should consist of real missile launches only. The probability of killing the primary target by the j^{th} missile launched ($j \leq M$, where M is the number of missiles the attacker has) is $(1-P)^{j-1} \cdot P$, and if that happens, the attacker's gain is $-jC_R+1$. The probability of not killing the target with all the M missiles is $(1-P)^M$, and in that case the payoff is $-MC_R$. Thus:

$$V_{M,0} = \sum_{j=1}^{M} (1-P)^{j-1} \cdot P[-jC_R+1] + (1-P)^M \cdot (-MC_R)$$

$$= -PC_R \cdot \sum_{j=1}^{M} j(1-P)^{j-1} + P \cdot \sum_{j=1}^{M} (1-P)^{j-1} - MC_R \cdot (1-P)^M.$$

Each of the two sums appearing in the last expression can easily be calculated. Carrying out this calculation with some further algebraic manipulation we finally find that:

$$V_{M,0} = (1 - \frac{C_R}{P})[1 - (1-P)^M] \qquad \text{(VI.16)}$$

We proceed by transforming the right hand side of Eq. (VI.14) to an explicit expression (see Appendix, Eq. (1)). After some more algebraic work, we are able to write the equation:

$$(V_{M,N} - V_{M,N-1})(C_R - P + V_{M,N} - (1-P)V_{M-1,N})$$

$$= C_D P(1-q) + C_D(1-P)V_{M-1,N} - C_D(1-Pq)V_{M-1,N-1} \,. \qquad \text{(VI.17)}$$

We define now:

$$\bar{V}_{M,N} = \frac{(1-\frac{C_R}{P}) - V_{M,N}}{1-\frac{C_R}{P}} = 1 - \frac{V_{M,N}}{1-\frac{C_R}{P}} \,. \qquad \text{(VI.18)}$$

We have found that $\bar{V}_{M,N}$ (which is dimensionless) is a very convenient quantity to work with mathematically. Besides, it has a very transparent and useful interpretation. To see this, notice that $1-\frac{C_R}{P}$ is the expected gain in a process in which the attacker is unlimited in the number of missiles, and in which no defense exists. By definition (VI.18), $\bar{V}_{M,N}$ measures the difference between the actual expected gain $V_{M,N}$ and the expected gain in that hypothetical, ideal case. This difference is given by $\bar{V}_{M,N}$ not in absolute terms, but rather in relative terms. If, for example, $\bar{V}_{M,N}$ is equal to -0.3, it indicates that the expected gain from the game $\Gamma^{M.N}$ is 30 percent higher than what would be expected if no defense existed and no limit was imposed on the number of the attacker's missiles. The value $\bar{V}_{M,N}$ is thus a very natural choice of a function with which a very meaningful and efficient dimensional analysis can be carried out.

Our next step is to write Eq. (VI.17) in terms of $\bar{V}_{M,N}$, instead of $V_{M,N}$. We find:

$$(\bar{V}_{M,N-1} - \bar{V}_{M,N})\,[(1-P)\bar{V}_{M-1,N} - \bar{V}_{M,N}] = \frac{C_R C_D (1-q)}{(1-\frac{C_R}{P})^2} + \frac{C_D(1-Pq)\bar{V}_{M-1,N-1}}{1-\frac{C_R}{P}} - \frac{C_D(1-P)\bar{V}_{M-1,N}}{1-\frac{C_R}{P}} .$$

We now define the parameter $f = C_R/P$, and again define:

$$r_c = \frac{C_D}{C_R} .$$

These parameters have very clear meanings. Substitution in the last equation yields:

$$(\bar{V}_{M,N-1} - \bar{V}_{M,N})\,[(1-P)\bar{V}_{M-1,N} - \bar{V}_{M,N}] = \frac{P^2 r_c (1-q) f^2}{(1-f)^2} - \frac{Pfr_c(1-P)}{1-f}\cdot\bar{V}_{M-1,N} + \frac{Pfr_c(1-Pq)}{1-f}\cdot\bar{V}_{M-1,N-1} . \quad \text{(VI.19)}$$

This equation is a difference equation, with the initial conditions:

$$\bar{V}_{0,N} = 1 \quad \text{(VI.18a)}$$

$$\bar{V}_{M,0} = (1-P)^M . \quad \text{(VI.18b)}$$

Eq. (VI.18) can be written as:

$$\bar{V}^2_{M,N} - B_{M,N}\cdot\bar{V}_{M,N} + C_{M,N} = 0 \quad \text{(VI.20)}$$

where:

$$B_{M,N} = (1-P)\cdot\overline{V}_{M-1,N} + \overline{V}_{M,N-1} \tag{VI.20a}$$

$$C_{M,N} = (1-P)\cdot\overline{V}_{M,N-1}\cdot\overline{V}_{M-1,N} - \frac{P^2f^2}{(1-f)^2}\cdot r_c(1-q)$$

$$+ \frac{Pfr_c}{1-f}(1-P)\overline{V}_{M-1,N} - \frac{fPr_c}{1-f}(1-Pq)\cdot\overline{V}_{M-1,N-1}\ . \tag{VI.20b}$$

and so:

$$\overline{V}_{M,N} = \frac{B_{M,N} \pm \sqrt{B^2_{M,N} - 4C_{M,N}}}{2} \tag{VI.21}$$

It remains yet to decide which of the two representations given in (VI.21) is the appropriate one. We first find the answer by looking at the case M = 1 (N arbitrary). This case is especially interesting; it corresonds to a case in which the attacker has a single, expensive (or espeically scarce) real weaon along with many cheap decoys. We shall give the solution $\overline{V}_{1,N}$, for all N, along with optimal strategies for both the defender and the attacker.

From (VI.20a) and (VI.20b) we have (since $\overline{V}_{0,1} = 1$):

$$B_{1,N} = 1-P + \overline{V}_{1,N-1}$$

$$C_{1,N} = (1-P)\overline{V}_{1,N-1} - \frac{P^2f^2}{(1-f)^2}r_c(1-q) + \frac{Pfr_c}{1-f}(1-P) - \frac{Pfr_c}{1-f}(1-Pq)$$

$$= (1-P)\overline{V}_{1,N-1} + \frac{Pfr_c}{(1-f)^2}\left[(1-P)(1-f) - Pf(1-q) - (1-Pq)(1-f)\right]$$

$$= (1-P)\overline{V}_{1,N-1} - \frac{P^2fr_c(1-q)}{(1-f)^2}$$

substituting in Eq. (VI.21) we get:

$$\bar{V}_{1,N} = \frac{(1-P+\bar{V}_{1,N-1}) \pm \sqrt{(1-P-\bar{V}_{1,N-1})^2 + 4\cdot\frac{P^2 fr_c(1-q)}{(1-f)^2}}}{2} \quad . \qquad \text{(VI.22)}$$

We show now that the correct sign in front of the square root is the plus sign. Suppose it were the minus sign. Then we would have, from Eq. (VI.22)

$$\bar{V}_{1,N} < \bar{V}_{1,N-1}$$

which implies

$$V_{1,N} > V_{1,N-1} \; .$$

This last inequality contradicts the obvious fact that $V_{1,N}$ (the expected payoff gained by the <u>attacker</u>) should <u>decrease</u> with N (the number of defensive missiles available). Thus:

$$\bar{V}_{1,N} = \frac{1-P+\bar{V}_{1,N-1} + \sqrt{(1-P-\bar{V}_{1,N-1})^2 + 4\cdot\frac{P^2 fr_c(1-q)}{(1-f)^2}}}{2} \quad . \qquad \text{(VI.23)}$$

For N = 0 we have $\bar{V}_{1,0} = 1-P$. So we can calculate $\bar{V}_{1,1}$:

$$\bar{V}_{1,1} = 1-P + \frac{P}{1-f}\cdot\sqrt{fr_c(1-q)} \; . \qquad \text{(VI.24)}$$

Notice that Eq. (VI.23) is valid only so long as there is no saddle point. We show that for such values of N, the values $\bar{V}_{1,N}$ can be written as:

$$\bar{V}_{1,N} = 1-P+K_N\left(\frac{P}{1-f}\right)\cdot\sqrt{r_c(1-q)f}$$

where K_N is a sequence of constants. Equation (VI.2- that this is true for N = 1, with $K_1 = 1$. We use ind to prove Eq. (VI.25) in general. Suppose it were true some N. We show that it is true for N+1 as well. Usi Eq. (VI.23) we find:

$$\bar{V}_{1,N+1} = \frac{1-P+\left(1-P+K_N\cdot\frac{P}{1-f}\cdot\sqrt{r_c(1-q)f}\right)}{2}$$

$$+\frac{1}{2}\sqrt{K_N^2\cdot\frac{P^2}{(1-f)^2}\cdot r_c(1-q)f+4\cdot\frac{P^2fr_c(1-q)}{(1-f)^2}}$$

$$= 1-P+\left(\frac{K_N}{2}+\sqrt{\left(\frac{K_N}{2}\right)^2+1}\right)\left(\frac{P}{1-f}\right)\cdot\sqrt{r_c(1-q)f}$$

$$= 1-P+K_{N+1}\left(\frac{P}{1-f}\right)\cdot\sqrt{r_c(1-q)f}$$

where

$$K_{N+1} = \frac{K_N}{2}+\sqrt{\left(\frac{K_N}{2}\right)^2+1}\ . \tag{VI.26}$$

The assertion is now proved. Eq. (VI.26) is a recursive relation for the sequence of constants $\{K_N\}$, (we start from $K_0 = 0$) appearing in the general expression for $\bar{V}_{1,N}$ given by Eq. (VI.25). From that equation and definition (VI.18) we find:

$$V_{1,N} = (1-\bar{V}_{1,N})(1-f) = P\left(1-f-K_N\cdot\sqrt{r_c(1-q)f}\right)\ . \tag{VI.27}$$

Eq. (VI.27) provides us with the complete solution to the exhaustion problem with $M = 1$. Once the parameters of the problem are given, it is a matter of straightforward calculation to find $V_{1,N}$ for any N, and from that, as we soon show, to calculate optimal strategies for both the defender and the attacker. We have to use the sequence of constants K_N which is given below for values of N from 1 to 16:

N	K_N
1	1
2	1.618
3	2.095
4	2.496
5	2.847
6	3.163
7	3.453
8	3.722
9	3.973
10	4.21
11	4.435
12	4.651
13	4.856
14	5.054

Table (Cont'd)

N	K_N
15	5.245
16	5.429

Using Eqs. (VI.20a), (VI.20b) and (VI.21), it is now a straightforward task to calculate all $\overline{V}_{M,N}$. It is impossible, however, to find analytic expressions for $M > 1$, since it seems hopeless to solve a difference equation of the type of Eq. (VI.21). Some attempts which we have made to replace Eq. (VI.21) by an approximating difference equation, failed to yield a tractable equation either. It is clear, however, that this does not pose any practical difficulties in computing the solution to any given pair of values of M and N, especially for values at ranges of practical interest, which are usually small.

Optimal Strategies.

We denote by $\pi^{of}(M,N)$ the probability--which is the function of the state (M,N)--with which the attacker chooses to launch a real missile when he uses his optimal strategy. We also denote by $\pi^{def}(M,N)$ the probability with which the defender chooses to fire when he behaves

optimally. Once all the values $V_{M,N}$ are given, the functions $\pi^{of}(M,N)$ and $\pi^{def}(M,N)$ can be calculated by directly applying the formulae in the Appendix. We apply those formulae to the matrix of the game $\Gamma^{M,N}$, as appears in Eq. (VI.15), and get:

$$\pi^{of}(M,N) = \frac{V_{M,N} - V_{M,N-1}}{-P(1-q)+(1-Pq)V_{M-1,N-1} + V_{M,N} - (1-P)V_{M-1,N} - V_{M,N-1}} \quad \text{(VI.29a)}$$

$$\pi^{def}(M,N) = \frac{V_{M,N} - C_D + C_R - P - (1-P)V_{M-1,N}}{-P(1-q) + (1-Pq)V_{M-1,N-1} + V_{M,N} - (1-P)V_{M-1,N} - V_{M,N-1}} \quad \text{(VI.29b)}$$

Equations (VI.29a) and (VI.29b) can be rewritten in terms of the normalized values and the parameters r_c, f, P, q as follows:

$$\pi^{of}(M,N) = \frac{\bar{V}_{M,N-1} - \bar{V}_{M,N}}{-P(1-q)\cdot\frac{f}{1-f} - (1-Pq)\bar{V}_{M-1,N-1} + \bar{V}_{M,N-1} - \bar{V}_{M,N} + (1-P)\,\bar{V}_{M-1,N}}$$

(VI.30a)

$$\pi^{def}(M,N) = \frac{-\frac{Pfr_c}{1-f} + (1-P)\bar{V}_{M-1,N} - \bar{V}_{M,N}}{-P(1-q)\cdot\frac{f}{1-f} - (1-Pq)\bar{V}_{M-1,N-1} + \bar{V}_{M,N-1} - \bar{V}_{M,N} + (1-P)\bar{V}_{M-1,N}}\,.$$

(VI.30b)

We use these formulae to calculate optimal strategies for the $\Gamma^{1,N}$ games (i.e., M = 1). The values $V_{1,N}$ and $V_{1,N-1}$ which appear here are substituted by their expressions given in Eq. (VI.25). We find:

$$\pi^{of}(1,N) = \frac{(K_N-K_{N-1})P\cdot\sqrt{r_c(1-q)\cdot f}}{P(1-q)+(K_N-K_{N-1})P\cdot\sqrt{r_c(1-q)\cdot f}}$$

$$= \frac{1}{1+\dfrac{\sqrt{\dfrac{(1-q)}{fr_c}}}{\sqrt{\left(\dfrac{K_{N-1}}{2}\right)^2+1}-\dfrac{K_{N-1}}{2}}} \qquad \text{(VI.31a)}$$

$$\pi^{def}(1,N) = \frac{Pfr_c+K_N P\cdot\sqrt{r_c(1-q)f}}{P(1-q)+(K_N-K_{N-1})P\cdot\sqrt{r_c(1-q)f}}$$

$$= \frac{1+\dfrac{1}{K_N}\cdot\sqrt{\dfrac{fr_c}{1-q}}}{1+\dfrac{1}{K_N}\left[\sqrt{\dfrac{1-q}{fr_c}}-K_{N-1}\right]} \qquad \text{(VI.31b)}$$

It should be emphasized that Eqs. (VI.31a) and (VI.31b), as well as Eq. (VI.25), are valid only for games which have no saddle points. We can quite easily find for which values of N the game $\Gamma^{1,N}$ does have a saddle point (so that the optimal strategies are pure). We do this by observing Eqs. (VI.31a) and (VI.31b). The right hand side of Eq. (VI.31a) can always represent a probability (it gives always values between 0 and 1), but the right hand side of Eq. (VI.31b) will represent a probability only if:

$$\sqrt{\frac{fr_c}{1-q}} < \sqrt{\frac{1-q}{fr_c}} - K_{N-1}$$

or when:

$$K_{N-1} < \sqrt{\frac{1-q}{fr_c}} - \sqrt{\frac{fr_c}{1-q}} \qquad \text{(VI.32)}$$

The sequence $\{K_N: N = 0,1,2,\}$ is an increasing sequence, since

$$K_N = \frac{K_{N-1}}{2} + \sqrt{\left(\frac{K_{N-1}}{2}\right)^2 + 1} > \frac{K_{N-1}}{2} + \frac{K_{N-1}}{2} = K_{N-1}$$

so that there is a number N^c defined by

$$N^c = \text{Max}\{N: \text{ Eq. (VI.32) is satisfied}\}. \qquad \text{(VI.33)}$$

The number N^c has the property that for all $N \leq N^c$, the optimal strategies of the game $\Gamma^{1,N}$ are randomized, that is, decoys should be deployed to gain optimal payoffs. For $N > N^c$ only real missiles should be used.

By squaring both sides of Eq. (VI.32) and solving for r_c, we can derive the following condition, which r_c should satisfy in order for the use of decoys to become beneficial at state (1,N):

$$r_c < \left(\frac{1-q}{f}\right) \cdot \left[1 + \frac{K_{N-1}^2}{2} - K_{N-1} \cdot \sqrt{\left(\frac{K_{N-1}}{2}\right)^2 + 1}\ \right]. \qquad \text{(VI.34)}$$

Saying it informally, Eq. (VI.34) tells how inexpensive a decoy should be (relative to the cost of a real missile), in order for it to be economical to use. It is also important to determine under what condition the decoys do not contribute to the overall effectiveness, at any value of N, or in other

words, when does N^c as defined in (VI.33) equal zero. Since $\{K_N\}$ is a monotone increasing sequence, we see that if r_c violates Eq. (VI.32) (hence also Eq. (VI.34)) for N = 1, it violates that equation for all N > 1 also. Therefore, putting N = 1 in Eq. (VI.34) noting that $K_0 = 0$, we get from Eq. (VI.34), that if

$$r_c > \frac{1-q}{f}, \tag{VI.35}$$

then decoys do not benefit the attacker in any case! This may be a very useful criterion to decide whether a decoy, as an exhausting device, should be acquired and held in the attacker's arsenal.

3. Numerical Example

The following numerical example illustrates the typical characteristics of the solutions to the model presented here. We assume:

$M = 1$ (attacker owns a single real missile)

$N = 1,2$

$q = 0.7$

$P = 0.5$

$C_R = 0.25$ (i.e., a real missile costs one quarter of the target 'worth').

We have also $f = C_R/P = 0.5$.

We present here the following results:

(a) The graphs of the values $\bar{V}_{1,1}$ and $\bar{V}_{1,2}$ as a function of r_c, the ratio between decoy cost and

real missile cost. These graphs are given in Fig. VI.9. As explained before, $\bar{V}_{1,N}$ $(N = 1,2)$ is a dimensionless quantity, expressing a relative measure of the difference between the expected gain (i.e., the value) in the actual game $(\Gamma^{M,N})$ and the expected gain in a hypothetical process in which the attacker attacks an undefended primary target, with no limit on the number of missiles he can use. We have:

$$\bar{V}_{1,N} = 1 - \frac{V_{1,N}}{1 - \frac{C_R}{P}} = \frac{(1 - \frac{C_R}{P}) - V_{1,N}}{1 - \frac{C_R}{P}} .$$

(Notice that $\bar{V}_N$ may be negative also. Being negative means that

$$V_{1,N} > 1 - \frac{C_R}{P}$$

that is, it indicates that the gain in the game $\Gamma^{1,N}$ is better than that in the hypothetical process described above.)

(b) The optimal policies for both the attacker and the defender are given in Fig. VI.10. They are characterized by π^{of} and π^{def}, the optimal probabilities of selecting, respectively, the real missile (by the attacker) and the firing decision (by the defender).

In Fig. VI.9 we notice that at some value of r_c (which depends on N, the values $\bar{V}_{1,N}$ shows discontinuity of slope. That

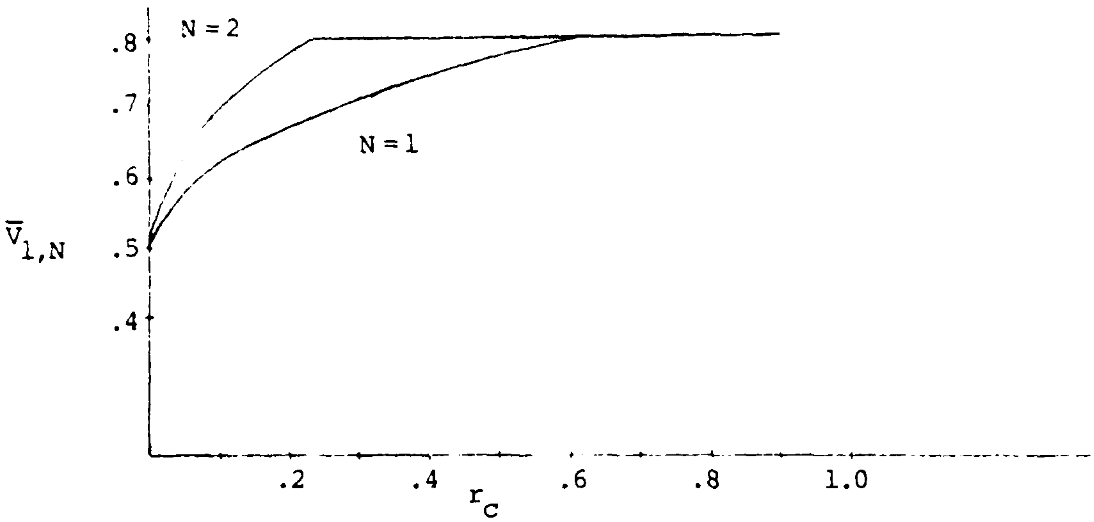


FIGURE VI.9: Dependence of the Modified Value $\overline{V}_{1,N}$ on the Decoy-To-Real Cost Ratio (RDA Game with Limited No. of Offensive Missiles)

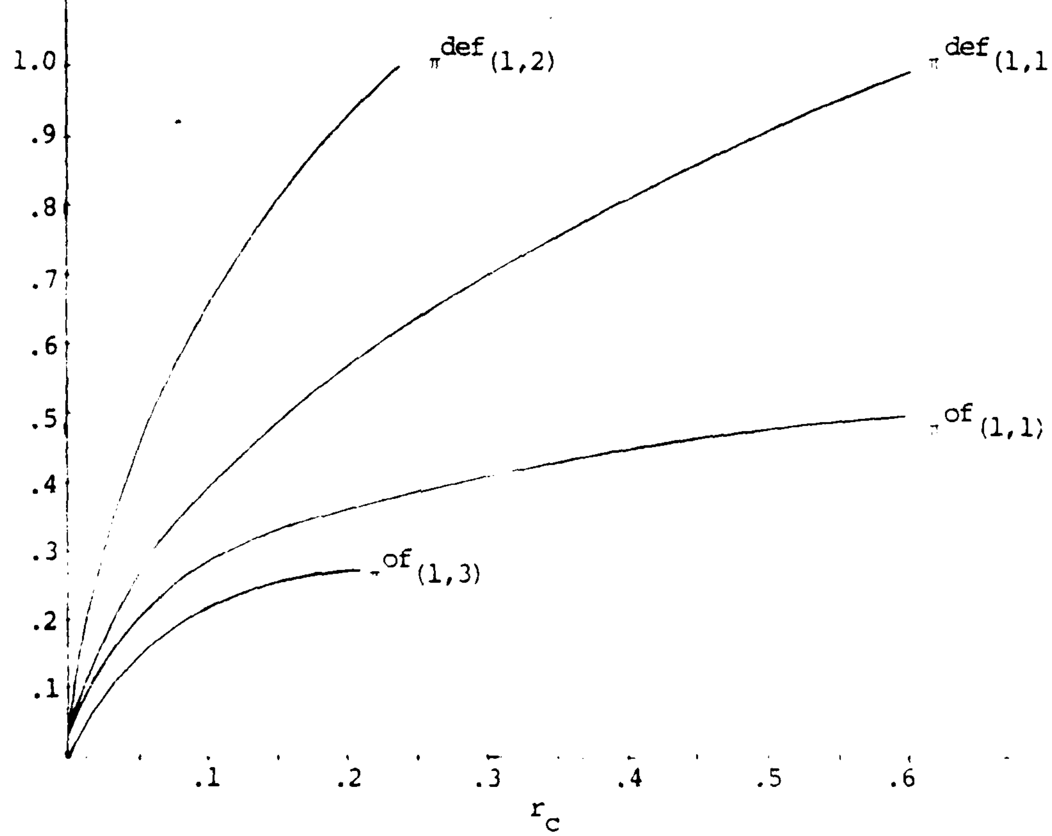


FIGURE VI.10: Optimal Strategies as Functions of Decoy-To-Real Cost Ratio (RDA Game with Limited No. of Offensive Missiles)

point is exactly the value of r_c, given in the right hand side of Ineq. (VI.34) (and for N = 1, the right hand side of (VI.35)). At that value of r_c, the optimal defensive and offensive policies cease to be randomized, and become pure policies. The value $\bar{V}_{1,N}$ thus cannot depend on r_c for values greater than the above value.

From Fig. VI.10 we notice that $\pi^{of}(1,1)$, $\pi^{of}(1,2)$, $\pi^{def}(1,1)$ and $\pi^{def}(1,2)$, all approach zero as $r_c \to 0$. We again have a situation in which no optimal policy exists although the value of the game does exist. If $r_c = 0$, the attacker has no motivation to stop using decoys--which costs nothing--since he anticipates that eventually the defender will consume his missiles. The defender, being aware that the attacker is bound to that kind of logic, has no reason to fire missiles. Thus the game is supposed to last forever with no gains earned by the attacker. It is quite simple to show however, that ε-optimal policies do exist for any $\varepsilon > 0$.

VII. OPTIMAL DEPLOYMENT OF DECOYS--MODELING THE SATURATION EFFECT

A. GENERAL CONCEPTS

The defense system is said to be in a state of saturation if the number of missiles arriving simultaneously at the borders of its killing zone is more than the number of missiles which the system is technically capable of engaging. Thus, the defense system has to select which missiles to engage, while all the other missiles are then able to penetrate the defense uninterrupted.

The situation we explore here is, again, one in which the detection of targets is made through a radar monitor only, on which real missiles and decoys produce signals with the same characteristics (i.e., intensity, radial velocity etc.). In other words, it is assumed that there is no way in which the radar operator can distinguish between a real missile and a decoy. This fact is exactly the one which the attacker is willing to exploit. By letting each real missile be accompanied by one or more decoys, the real missile is protected simply by the fact that there is a given prior probability that it will not be chosen for engagement by the defense. The more decoys accompanying the real missile, the more likely it is that it will not be engaged.

In Section B of this chapter we present and solve the problem of finding the optimal number of decoys to accompany

a real missile, in order to minimize the expected cost of destroying the primary target. It is clear that by launching more decoys the protection of the real missile becomes more effective. On the other hand, spending more decoys also increases the cost of the operation. We are looking for the optimal number of decoys, where the optimum represents a point of "balance" between the degree of protection and the cost.

In Section C we consider a similar problem, differing from that of Section B in that we make the number of real missiles also unspecified. In that case, the attacker can launch any mixture of real missiles and decoys, where neither the number of missiles of each type, nor their proportion are subject to any constraint.

The main purpose of deploying decoys, in any situation, is to improve the cost-effectiveness ratio of a given operation. In both models presented in this chapter we assume that the attacking processes are to continue until the primary target is destroyed. The objective is thus the expected cost of destroying the target.

B. THE OPTIMAL NUMBER OF DECOYS REQUIRED TO PROTECT A SINGLE REAL MISSILE

Suppose the attacker is restricted to launch only <u>one</u> real missile at a time. This restriction may originate with some technical or other type of constraints. However, there is no limit on the number of decoys the attacker is allowed to launch

simultaneously with the real missile. We assume that the attacker launches a "wave" of objects, one of them being the real missile. He then receives the information on whether the real missile hit the primary target or not. The primary target is defended by N_s secondary targets. We use the following notation:

- m_D - No. of decoys shot simultaneously with a real missile, at each "wave" of attack
- C_R - Cost of a real missile
- C_D - Cost of decoy ($C_D < C_R$)
- C^o - Optimal expected cost of destroying the primary target
- P - Probability that the real missile kills the target, given that it survives
- q - Probability that the real missile survives a given interception attempt made by any single defense target
- N_s - No. of secondary targets defending the primary target.

We also make the following assumptions:

(1) One real missile is launched together with m_D decoys. The attacker controls the launches in such a way as to let all the objects reach the boundaries of the defense killing zone at about the same time. Consequently, each secondary target is capable of engaging exactly one missile.

(2) The operators of the detection radars sitting at the defense targets are incapable of distinguishing between a real missile and a decoy.

(3) It is assumed that the attack process will go on until the primary target is killed. After each attack in which m_D+1 missiles (one real and m_D decoys) are used, the attacker is informed about tne result. He quits immediately upon achieving a hit of the primary target.

(4) Two cases concerning the engagement process are considered:

(a) Case 1: In this case each one of the N_s secondary targets operates independently of all the others. The selection of the missile to be engaged is made randomly, each one of the m_D+1 missiles has an equal chance of being selected by each of the secondary targets. Thus it is possible that some of the missiles will be engaged by more than one secondary target, whereas others will be engaged by none!

(b) Case 2: In this case the operations of the N_s secondary targets are coordinated so that the resulting engagement pattern is of a one-on-one type. One way to visualize that is to assume that there is a central unit equipped with its own means of target detection and control, and having command links with all the N_s secondary targets. This unit is responsible for distributing the targets among the subordinate launching units (i.e., the secondary targets) in such a way that no more than one secondary target is assigned to engage any single detected object. For the saturation effect to exist we have to require that $N_s < m_D+1$. (Otherwise, if $N_s \geq m_D+1$, each missile (including the real one) will be engaged with probability one so that no condition of saturation really exists.) It is further assumed that any group of N_s objects (out of the

$$\binom{m_D+1}{N_s}$$

different groups) has an equal chance of being selected as the group of engaged missiles.

1. Solution to Case 1: Independent Operations of the Secondary Targets

We first calculate the probability that the real missile will survive given that it is accompanied by m_D decoys. To do that, notice that the probability that any given secondary target will intercept the real missile is given by:

$$\Pr\left\{\begin{matrix}\text{The real missile will}\\ \text{be engaged by the target}\end{matrix}\right\} \cdot \Pr\left\{\begin{matrix}\text{The real missile will be}\\ \text{intercepted if engaged}\end{matrix}\right\}$$

$$= \frac{1}{m_D+1} \cdot (1-q) \ .$$

The probability of surviving the N_s secondary targets is thus

$$\left(1 - \frac{1-q}{m_D+1}\right)^{N_s} .$$

The probability that the real missile will not kill the primary target is therefore:

$$P_r(\text{missing the primary}) = 1 - P \cdot \left(1 - \frac{1-q}{m_D+1}\right)^{N_s} \qquad \text{(VII.1)}$$

We now write the equation for C^o, the minimal expected cost of destroying the primary target. As a result of each ripple launch of the m_D+1 missile, two outcomes are possible:

(1) The primary target is killed, and so the process ends with no more cost incurred.

(2) The primary target is not killed, in which case the attacker remains at the same state he was before the launch, i.e., the optimal expected cost left to be paid is exactly C^o.

Thus, the attacker has to pay the cost of the missiles he launches at any single stage, plus the expected cost of continuing the process, which is either 0 (in Case (1) above) or C^o (in Case (2)). The probability of Case (2) is given in Eq. (VII.1). Therefore we have the following equation:

$$C^o = \min_{\substack{m_D \geq 0 \\ (\text{integer})}} \left\{ m_D C_D + C_R + \left[1 - P \cdot \left(1 - \frac{1-q}{m_D+1}\right)^{N_s}\right] \cdot C^o \right\} . \qquad \text{(VII.2)}$$

Notice that C^o, the unknown optimal value of the objective function, appears on both sides. This equation can be solved as follows. First, denote by $C(m_D)$ the expected cost of

destruction when there are m_D decoys accompanying each real missile. Then:

$$C(m_D) = m_D C_D + C_R + [1 - P(1 - \frac{1-q}{m_D+1})^{N_s}]C(m_D)$$

or

$$C(m_D) = \frac{m_D \cdot C_D + C_R}{P\cdot(1 - \frac{1-q}{m_D+1})^{N_s}} \quad \text{(VII.3)}$$

Now calculate the derivative of $C(m_D)$ (disregarding, for the moment, the fact that m_D is confined to be an integer):

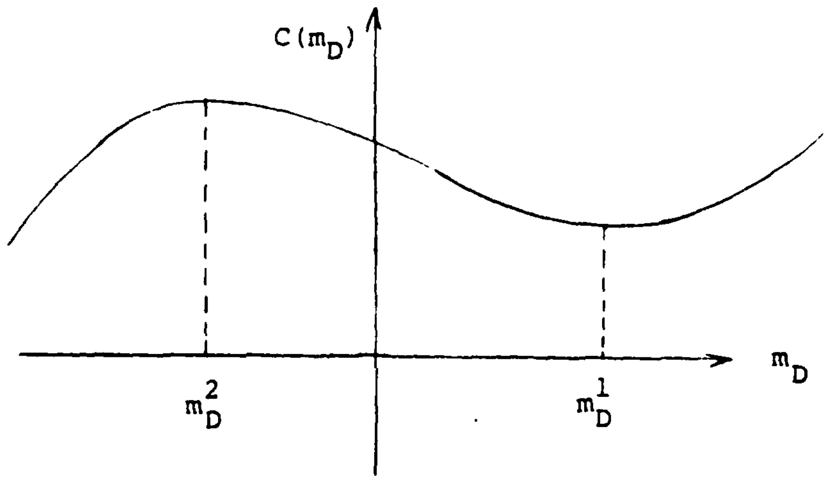
$$\frac{dC(m_D)}{dm_D} = \frac{C_D\cdot(1 - \frac{1-q}{m_D+1})^{N_s} - (m_D C_D + C_R)N_s(1 - \frac{1-q}{m_D+1})^{N_s-1}\cdot\frac{1-q}{(1+m_D)^2}}{P(1 - \frac{1-q}{m_D+1})^{2N_s}}$$

$$= \frac{1}{P(1 - \frac{1-q}{m_D+1})^{N_s+1}(m_D+1)^2}\cdot[C_D m_D^2 + C_D[q+1-N_s(1-q)]m_D + C_D q - C_R N_s(1-q)] .$$

It can be seen that as the variable m_D goes to either $+\infty$, or $-\infty$, the derivative of $C(m_D)$ converges to a positive value C_D/P. Also, there are exactly two points at which this derivative becomes zero. They are the two roots of the quadratic equation:

$$C_D\cdot m_D^2 + C_D[q+1-N_s(1-q)]\cdot m_D + C_D q - C_R N_s(1-q) = 0 . \quad \text{(VII.4)}$$

Let m_D^1 and m_D^2 be the two roots of this equation. If one root is negative and the other is non-negative, then the function $C(m_D)$ will have the following (tentative) graphical from (m_D^1 being possibly zero):



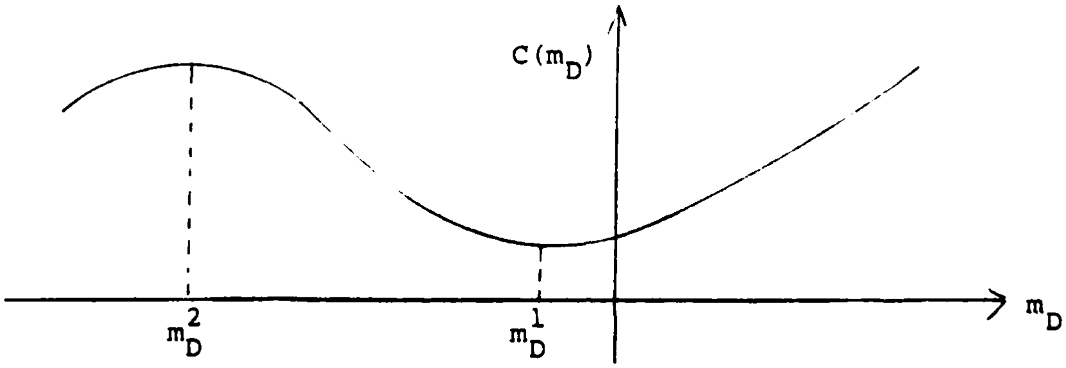
Let m_D^1 be the positive root of Eq. (VII.4). It can be seen from the above discussion that the solution to the actual problem (where m_D is constrained to be a non-negative integer) is:

$$m_D^o = \begin{cases} [m_D^1] & \text{if } C([m_D]) \leq C([m_D]+1) \\ [m_D^1]+1 & \text{if } C([m_D]) > C([m_D]+1) \end{cases} \qquad \text{(VII.5)}$$

(where $[\cdot]$ denotes, as usual, the integral part). We have also:

$$C^o = \mathrm{Min}\{C([m_D^1]), C([m_D^1]+1)\}$$

If both roots are non-positive then the function $C(m_D)$ has the form



In this case the solution is simply $m_D^o = 0$, and $C^o = C(0)$, since m_D is restricted (in the actual problem) to the non-negative integers.

A third case which should, theoretically, be considered is the one where both m_D^1 and m_D^2 are non-negative. This case will soon be shown to be impossible for this problem.

We now give the explicit solution to Eq. (VII.4) through which we will be able to derive conditions for each of the two cases mentioned above to occur.

We have from (VII.4)

$$m_D^{\substack{1\\2}} = \frac{1}{2}\left[N_s(1-q) - (1+q) \pm \sqrt{[N_s(1-q)-(1+q)]^2+4\left(\frac{N_s(1-q)}{r_c} - q\right)}\right] \qquad \text{(VII.8)}$$

where $r_c = C_D/C_R$ is the decoy-to-real cost ratio. We now write the conditions for each of the two cases mentioned above:

(a) The condition for one root to be negative and for the other one to be non-negative is:

$$\frac{N_s(1-q)}{r_c} - q \geq 0$$

or

$$q \leq \frac{N_s}{N_s + r_c} .$$

(b) The conditions for both roots to be negative are:

$$N_s(1-q) - (1+q) < 0 \qquad q > \frac{N_s - 1}{N_s + 1} \tag{VII.9a}$$

and

$$\frac{N_s(1-q)}{r_c} - q < 0 \qquad q > \frac{N_s}{N_s + r_c} . \tag{VII.9b}$$

Since $r_c < 1$ we have

$$\frac{N_s}{N_s + r_c} > \frac{N_s - 1}{N_s + 1} , \tag{VII.9c}$$

and so, condition (VII.9a) is redundant, and we have

$$q > \frac{N_s}{N_s + r_c}$$

as a single condition under which the optimal value of m_D is $m_D^o = 0$.

We now show that it is impossible that both m_D^1 and m_D^2 of Eq. (VII.8) will be non-negative. The conditions for that should have been:

$$N_s(1-q) - 1+q) \geq 0 \Longrightarrow q \leq \frac{N_s-1}{N_s+1}$$

and

$$\frac{N_s(1-q)}{r_c} - q \leq 0 \Longrightarrow q \geq \frac{N_s}{N_s + r_c} .$$

Both of these two conditions cannot be satisfied since from inequality (VII.9c), we see that they contradict each other. Thus, this case is vacuous.

We now summarize the algorithm which solves Case 1 (independent operations of the defense targets):

(a) Calculate m_D^1 (Eq. (VII.8), taking the plus sign in front of the square root).

(b) There are two possibilities regarding the optimal number of decoys to accompany a real missile:

(b1) If $q > (N_s)/(N_s + r_c)$, the solution is $m_D^o = 0$. It means that if the probability of survival of the real missile--given that it is engaged--is "sufficiently high" (and to be specific, higher than $N_s/(N_s + r_c)$, the attacker should not resort to decoys, since their contribution to the survivability is not worth their cost.

(b2) If $q \leq N_s/(N_s + r_c)$, the solution is $[m_D^1]$ or $[m_D^1]+1$, depending on whether $C([m_D^1])$ is less than or greater than $C([m_D^1]+1)$. The function $C(m_D)$ can be written as (see Eq. (VII.3)):

$$C(m_D) = \frac{f(1+m_D r_c)(1+m_D)^{N_s}}{(m_D+q)^{N_s}} .$$

To make significant quantitative analysis of the improvements achieved by introducing decoys, we should measure the expected cost of destruction relative to some standard unit of cost. The most relevant basis for that purpose is the expected cost of destruction when using real missiles only, which clearly is $C_R/Pq^{N_s} = f/q^{N_s}$. We still subject ourselves to the requirement that only one real missile may be launched at each attempt. (We drop this constraint in Section C, where we explore the problem of optimal real/decoy "mixtures"). We now define the dimensionless function $\bar{C}(m_D)$.

$$\bar{C}(m_D) = \frac{C(m_D)}{f} \cdot q^{N_s} = (1 + m_D r_C)\left(\frac{q+qm_D}{q+m_D}\right)^{N_s} \qquad \text{(VII.10)}$$

It is apparent from the discussion above that the value $\bar{C}^o = \bar{C}(m_D = m_D^o) = \frac{C^o q^{N_s}}{f}$ directly indicates by how much do the decoys contribute to the reduction of the expected cost of destruction.

2. <u>Solution to Case 2: Coordinated Operations of the Secondary Targets</u>

In this case we assume that the operations of the individual defense targets are coordinted and that no missile is engaged by more than one secondary target. We assume so even when the number of missiles the attacker launches is less than or equal to the number of secondary targets. Therefore, if the number of secondary targets is N_s, then the optimal number of decoys (m_D) can be either greater than N_s-1 (so that together with the real missile there will be more than N_s missiles), or it can be zero, so that the real missile

will be launched alone. The third possibility, namely that $0 < m_D \leq N_s - 1$, is surely non-optimal since under the coordinated operations assumptions there will be (with probability one) a secondary target which will engage the real missile. Thus, as long as there are not more than N_s-1 decoys accompanying the real missile, no protection is provided by the decoys, although they do add to the cost. Therefore, choosing $m_D = 0$ is always preferable to choosing any number between 1 and N_s-1.

If $m_D+1 > N_s$, there are $\binom{m_D+1}{N_s}$ different ways in which N_s missiles can be selected to be engaged. We assume that each group of N_s missiles has equal chance of being selected as the group of engaged missiles. Since there are $\binom{m_D}{N_s-1}$ groups of N_s objects, one of which is the real missile, we find that the probability that the real missile will be engaged is:

$$\Pr\{\text{Real missile engaged}\} = \frac{\binom{m_D}{N_s-1}}{\binom{m_D+1}{N_s}} = \frac{N_s}{m_D+1} .$$

The probability that the real missile kills the primary target is thus:

$$P\left[1 - \frac{N_s}{m_D+1}(1-q)\right] .$$

The equation for C^o in this case is therefore:

$$C^{o} = \text{Min}\begin{cases} \displaystyle\min_{\substack{m_D \geq N_s \\ \text{(integer)}}} \left\{ m_D C_D + C_R + \left[1-P\left(1-\frac{(1-q)N_s}{m_D+1}\right)\right] C^{o} \right\} , \\ \dfrac{C_R}{Pq} \end{cases} \qquad \text{(VII.11)}$$

The term in the lower row reflects the possibility that the optimal value of m_D is zero. The upper term corresponds to all other cases, and as was explained above, if m_D is non-zero, it must be greater than N_s-1, so that the minimum in the upper term is taken over all m_D greater than or equal to N_s.

As before, we are interested in a dimensionless quantity representing cost, which provides a convenient quantitative measure of the contribution of decoys to cost reduction. In this case this quantity is $\bar{C}^{o}$ defined by:

$$\bar{C}^{o} = \frac{C^{o}}{C_R/Pq} .$$

The value $\bar{C}^{o}$ gives the cost of destruction relative to the cost that would have been paid if decoys had not been allowed to be deployed. Using $\bar{C}^{o}$ instead of C^{o}, Eq. (VII.11) becomes:

$$C^{o} = \text{Min}\begin{cases} \displaystyle\min_{\substack{m_D \geq N_s \\ m_D \text{ integer}}} \left\{ Pq(m_D r_c+1) + \left[1-P\left(1-\frac{(1-q)N_s}{m_D+1}\right)\right] \bar{C}^{o} \right\} . \\ 1 \end{cases} \qquad \text{(VII.12)}$$

We proceed by solving Eq. (VII.12), following logical steps similar to those made in solving Eq. (VII.2). The only difference is that now we have different forms for the functions $C(m_D)$ and $\bar{C}(m_D)$, which for the case of independent operations were given in Eqs. (VII.3) and (VII.10), respectively.

Here we define:

$$C(m_D) = \frac{m_D \cdot C_D + C_R}{P\left[1 - \frac{(1-q)N_s}{m_D+1}\right]} \quad ,$$

$$\bar{C}(m_D) = \frac{C(m_D)}{C_R/Pq} \equiv q \cdot \frac{(m_D r_C + 1)(m_D+1)}{m_D+1-(1-q)N_s} \quad . \qquad \text{(VII.13)}$$

The solution m_D^o to Eq. (VII.12) should be either zero, or the one which minimizes $\bar{C}(m_D)$, if this minimizing value is greater than N_s-1. First we calculate this minimum of $\bar{C}(m_D)$, temporarily ignoring the integer constraint. Differentiating $\bar{C}(m_D)$ we find:

$$\frac{d\bar{C}(m_D)}{dm_D} = q\frac{[2r_C m_D + r_C + 1][m_D + 1 - (1-q)N_s] - (m_D r_C + 1)(m_D + 1)}{[m_D + 1 - (1-q)N_s]^2}$$

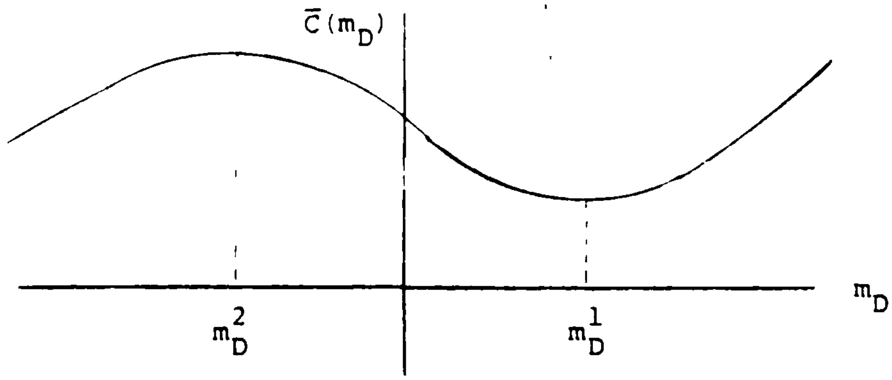
$$= q \cdot \frac{r_C m_D^2 + 2r_C(1-(1-q)N_s)m_D + (r_C+1)(1-(1-q)N_s) - 1}{[m_D + 1 - (1-q)N_s]^2}$$

The solution to the equation $d\bar{C}(m_D)/dm_D = 0$ is

$$m_D^{1,2} = \frac{[(1-q)N_s - 1] \pm \sqrt{[(1-q)N_s-1]^2 - (1+\frac{1}{r_C})(1-(1-q)N_s) + \frac{1}{r_C}}}{2} \qquad \text{(VII.14)}$$

From definition (VII.13) it is easily seen that as $m_D \to \pm\infty$, the function $\bar{C}(m_D)$ approaches a linear increasing curve, with positive slope. Hence the root which corresponds to the minimum is the larger one (m_D^1, the root with the plus sign). In

general, m_D^1 will be non-integer. The following figure shows the general form of the graph of $\bar{C}(m_D)$.



In order to find the solution to the actual problem (i.e., with the integer constraint imposed), we must consider three different cases:

(a) If $m_D^1 \geq N_s$, we define, for any m_D^1

$$\bar{C}_{min}(m_D^1) = \text{Min}\{1, \bar{C}([m_D^1]), \bar{C}([m_D^1]+1)\}$$

From the discussion above it is seen that m_D^o can be expressed as:

$$m_D^o = \begin{cases} [m_D^1] & \text{if } \bar{C}_{min}(m_D^1) = C([m_D^1]) \\ [m_D^1]+1 & \text{if } \bar{C}_{min}(m_D^1) = C([m_D^1]+1) \\ 0 & \text{if } \bar{C}_{min}(m_D^1) = 1 \end{cases}$$

(b) If $m_D^1 < N_s$, then the optimum is either $m_D^o = N_s$ or $m_D^o = 0$, depending upon whether $\bar{C}(N_s)$ is less than (or equal to) $\bar{C}(D) = 1$, or is greater than 1. This is so because if either $[m_D^1]$ or $[m_D^1]+1$ is positive and less than (or equal) to

N_s-1, it cannot be the optimum by the argument given before (i.e., that no saturation effect is created). If, for instance, $N_s > 2$, and m_D^1 is such that

$$1 \leq m_D^1 < N_s - 1$$

then neither $[m_D^1]$ nor $[m_D^1]+1$ can be the optimal value m_D^1. Thus, in this case we have:

$$m_D^o = \begin{cases} N_s & \text{if} \quad \bar{C}(N_s) \leq 1 \\ \\ 0 & \text{if} \quad \bar{C}(N_s) > 1 \end{cases} .$$

A remark should be made here that it is quite simple to derive a sufficient condition for decoys to be not worthy of their cost. From the graph of $\bar{C}(m_D)$ given above, we notice that if m_D^1 and m_D^2 are both non-positive, then the minimum of $\bar{C}(m_D)$ on the feasible domain (m_D = non-negative integer) should occur at $m_D = 0$. Thus the solution to Eq. (VII.12) is necessarily $m_D^o = 0$. From Eq. (VII.14), sufficient conditions for that to occur are:

$$(1 + \frac{1}{r_c})(1-(1-q)N_s) + \frac{1}{r_c} \leq 0 \Longrightarrow q \geq 1 - \frac{r_c}{N_s(1+r_c)} \qquad \text{(VII.15a)}$$

$$(1-q)N_s - 1 \leq 0 \longrightarrow q \geq 1 - \frac{1}{N_s} . \qquad \text{(VII.15b)}$$

Since $0 < r_c < 1$, the existence of the first condition guarantees the existence of the second, hence the second is

redundant and we conclude that a sufficient (although clearly not necessary) condition for decoys to be uneconomical to use is:

$$q \geq 1 - \frac{r_c}{N_s(1 + r_c)} . \qquad \text{(VII.16)}$$

3. <u>Numerical Example</u>

In order to gain some feeling about the character of the solutions given in Section B and about actual numbers of decoys required to operate optimally in some particular cases, we present a numerical example. Through all the following numerical cases it is assumed that P = Probability of kill (given survival) = 0.5. We have calculated m_D^o (= optimal number of decoys) and $\bar{C}(m_D^o)$, the optimal normalized cost, for various combinations of the parameters r_c and q.

The results are presented in the format shown in Figs. VII.1-VII.3. In Fig. VII.1, optimal values for Case I (independent operations) are given when N_s = number of secondary targets = 3. The optimal number of decoys m_D^o for any combination of r_c and q is readily given by observing the zone to which the point (r_c, q) belongs. The various zones are defined by a set of curves as shown. We have restricted our calculations to values of r_c between 0.1 and 0.9 (decoys which cost less than one tenth of the real missile, or more than nine tenths are rarely realistic anyway). From a pure theoretical point of view, however, it is obvious that all curves should converge to $q = 1$, as $r_c \to 0$, since when $r_c = 0$ (i.e.,

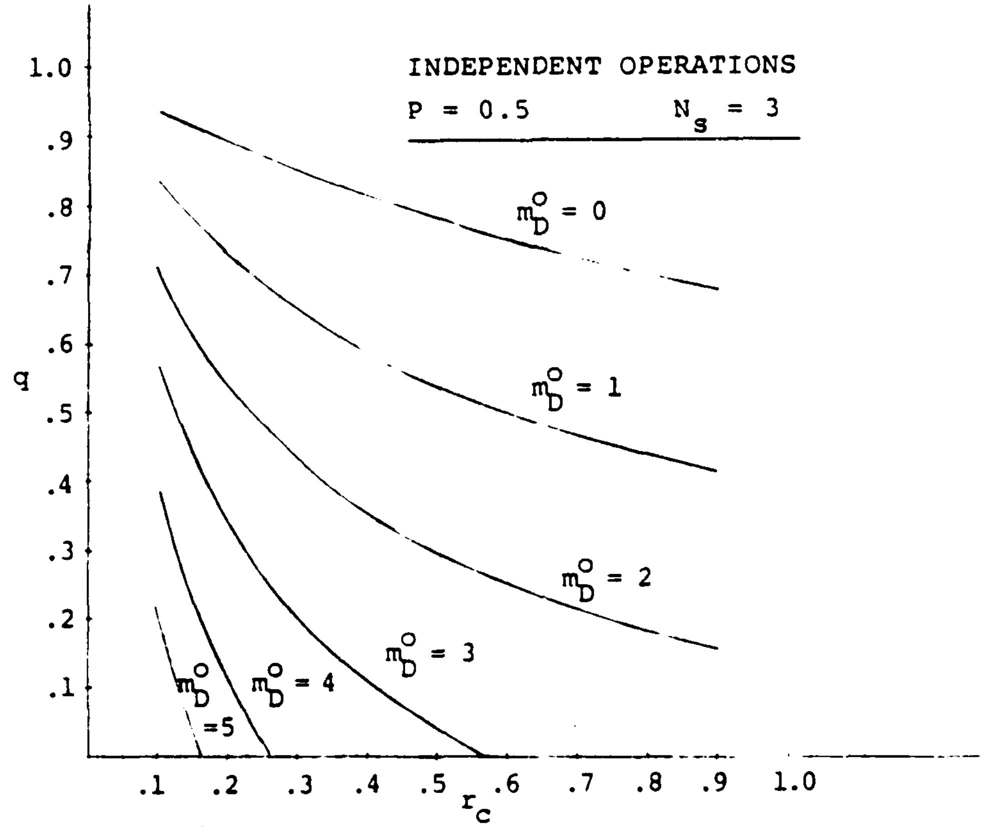


FIGURE VII.1: Optimal No. of Decoys Required to Protect a Single Real Missile--Case I (Independent Operations of Secondary Targets, N_s = 3)

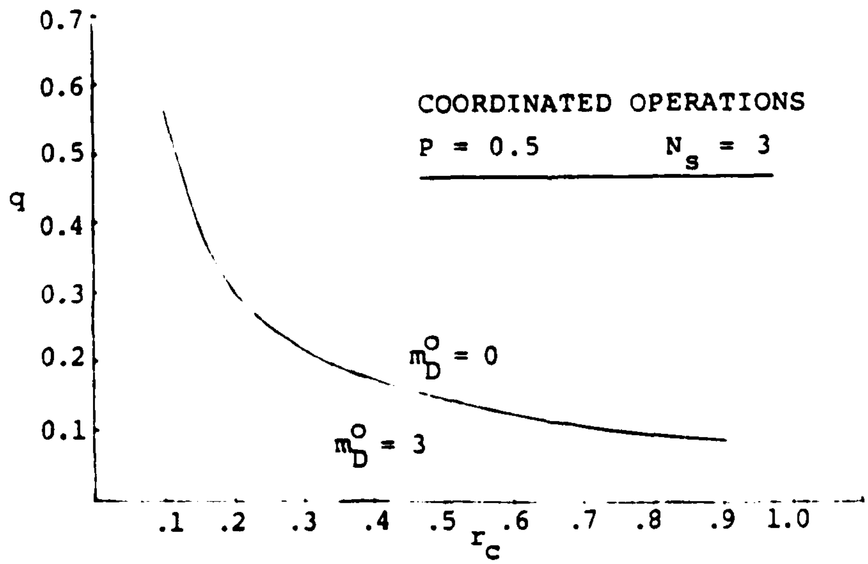


FIGURE VII.2: Optimal No. of Decoys Required to Protect a Single Real Missile--Case II (Coordinated Operations of Secondary Targets, N_s = 3)

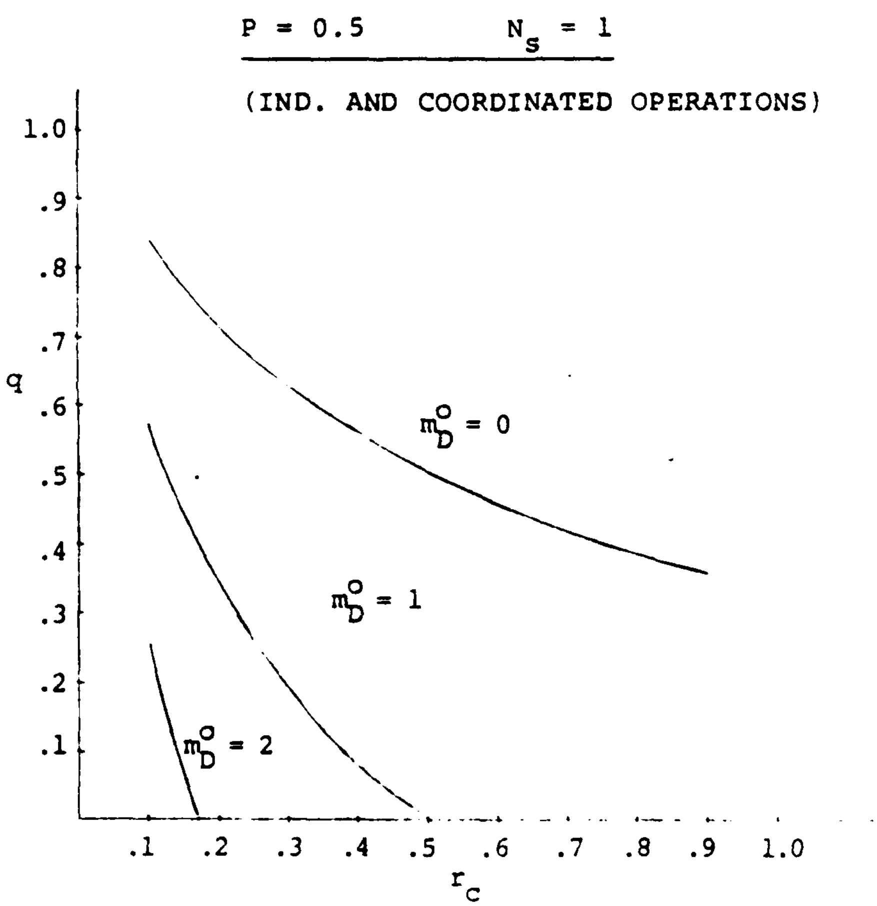


FIGURE VII.3: Optimal No. of Decoys Required to Protect a Single Real Missile ($N_s = 1$)

when decoys cost nothing!) any number of decoys is worth using, and the more we use the less is the expected cost of destruction. On the other hand, as $r_c \to 1$, the whole problem becomes irrelevant to any realistic situation. This is so mainly because the requirement that only one real missile be launched at any single stage is absurd: If r_c is very close to 1, it is obviously better to use several real missiles than to launch decoys, which cost almost the same as real missiles but contribute nothing to the probability of kill.

In Fig. VII.2 the results correspond to Case II (coordinated operations), with $N_s = 3$, as before, It is noteworthy that in most of the area of the square $\{(r_c,q:$ $0 \le r_c \le 1, 0 \le q \le 1\}$ optimal number of decoys is either three (so that decoys are added to create an excess of just one missile over what the defense can simultaneously handle) or zero. A comparison between Case I and Case II, for $N_s = 3$, reveals that there is a large area for which the optimal policy for Case II does not use decoys (i.e., $m_D^o = 0$) whereas the optimal policy for Case I uses one, two or even three decoys. We also observe that the maximal survival probabilities which requires the use of any given number of decoys is always smaller in Case II than in Case I (for any value of r_c). For example: If $r_c = 0.5$, it requires that the probability of survival be at most 0.145 in Case II in order that the optimal behavior will be to launch three decoys along with the real missile. In Case I, the probability of survival should be at most 0.295 in order for the same conclusion to hold.

In Fig. VII.3 we present the results for $N_s = 1$. Obviously, if only one secondary target is assumed to exist, there should be no difference between Case I (independent operations) and Case II (coordinated operations).

Figs. VII.4 and VII.5 present the optimal cost $\bar{C}^o = \bar{C}(m_D^o)$ for Case I (independent operations of secondary targets) as a function of r_c for various values of q (q = 0.2, 0.4, 0.6). Each set of three curves corresponds to either of the two cases analyzed in Section B and to a given pair of values of P and N_s (P = 0.5, N_s 1,3). Notice that $\bar{C}(m_D^o) = \bar{C}^o$ is a dimensionless quantity which gives the optimal cost of destruction when decoys are available, measured in terms of the optimal cost when decoys are unavailable. $\bar{C}^o$ is thus the natural measure of the overall effectiveness of decoys. Having $\bar{C}^o$ equal 0.2 for example, means that decoys make it possible to reduce the cost of destruction to only one fifth of what it would cost to destroy the target by the real missile only.

Note that the graphs of $\bar{C}^o$ as a function of r_c (for any given q, **values** of the parameters q and N_s) are piecewise linear. That this should be so may be derived directly from Eqs. (VII.10) and (VII.13), which give the form of $\bar{C}(m_D)$ for the two cases analyzed in Section B. From that equation we see that for a given m_D, the quantity $C(m_D)$ is a linear function of r_c. The points on the graphs where discontinuities of slope occur, are exactly those where the optimal number of decoys (m_D^o) changes.

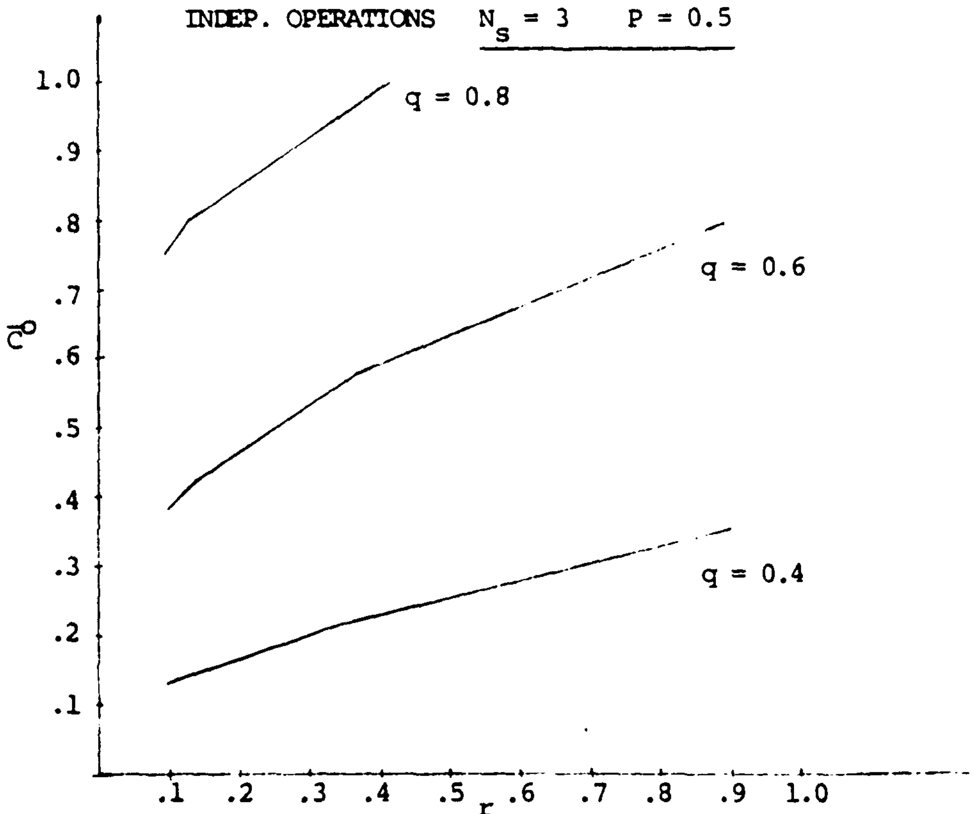


FIGURE VII.4: Optimal Expected Cost-of-Destruction Curves (N_s = 3, P = 0.5, Indep. Operations of Secondary Targets)

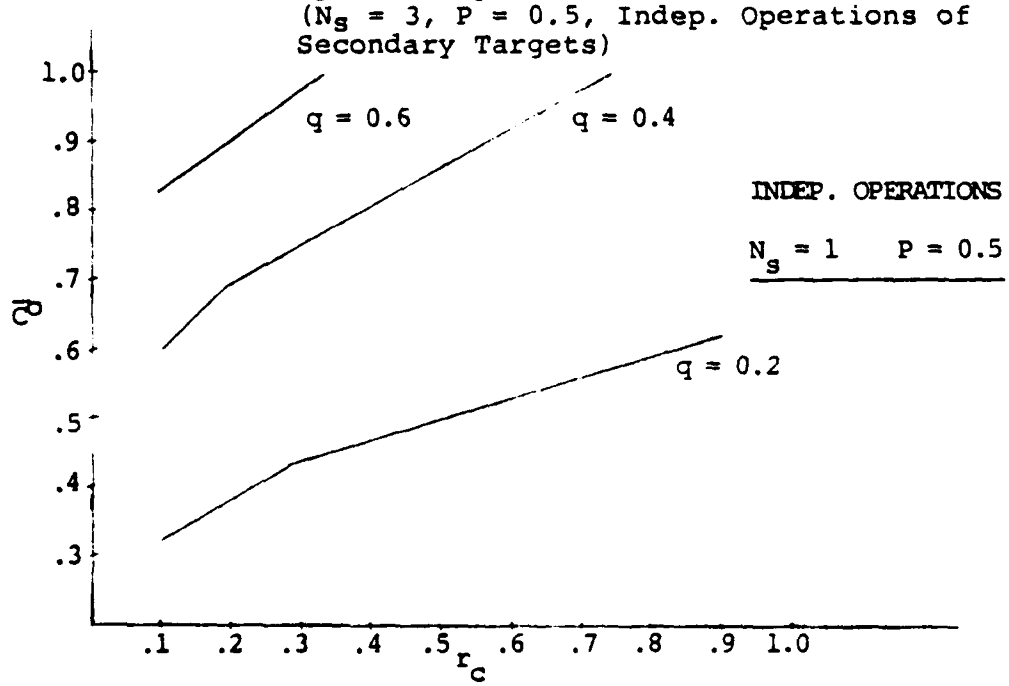


FIGURE VII.5: Optimal Expected Cost-of-Destruction Curves (N_s = 1, P = 0.5, Indep. Operations of Secondary Targets)

C. OPTIMAL MIXTURE OF REAL MISSILES AND DECOYS

1. General Solution

In Section B it was assumed that only one real missile is launched at each stage. While this may be a very realistic constraint, especially when the real missile is very expensive or scarce, it does not always apply. Therefore it is of special interest to find optimal "mixture" of real missiles and decoys when the number of real missiles launched at any given stage is not limited to one. We shall assume that only one secondary target is present, and that the objective function is (as it was in Section B) to minimize the expected cost of destroying the primary target.

Let m_R and m_D be the number of real missiles and decoys, respectively, launched simultaneously by the attacker. The parameter P is, as before, the single shot probability of kill of the primary target by a real missile (given that it survives). The parameter q is the probability of survival of the real missile given that it is engaged. We assume that the defense target is capable of engaging only one missile out of each "wave", and that each of the $m_D + m_R$ missiles has an equal chance of being selected for engagement. Thus there are two possibilities:

(a) That all the m_R missiles are left unengaged. The probability of this is $m_D/(m_D+m_R)$. The probability that they will all miss the primary target is $(1-P)^{m_R}$.

(b) That one of the real missiles will be selected by the defense for engagement. Thus only m_R-1 real missiles will be free to penetrate. Each has a probability of

miss which is equal to 1-P. The engaged real missile will have a probability of miss equal to 1-Pq. The probability for that event to occur is $m_R/(m_D+m_R)$.

We conclude that the probability that a group of m_D decoys and m_R real missiles, launched simultaneously, will miss the primary target is:

$$P_{miss}(m_R, m_D) = \frac{m_D}{m_R+m_D}\cdot(1-P)^{m_R} + \frac{m_R}{m_D+m_R}\cdot(1-P)^{m_R-1}\cdot(1-Pq)$$

$$= (1-P)^{m_R-1}\left[1-P+\frac{m_R}{m}\cdot P\cdot(1-q)\right] \qquad \text{(VII.17)}$$

where we have written

$$m = m_R + m_D .$$

Let $C^o(m)$ be the minimal expected cost of destruction when the attacker is constrained to launch a total of m missiles at a time. Thus we have:

$$C^o(m) = \min_{\substack{1\le m_R\le m \\ m_R \text{ integer}}} \{C_R m_R + C_D(m-m_R) + P_{miss}(m_R, m-m_R)\cdot C^o(m)\}$$

$$= C_D\cdot m + \min_{\substack{1\le m_R\le m \\ m_R \text{ integer}}} \{(C_R-C_D)m_R + (1-P)^{m_R-1}\cdot\left[1-P+\frac{m_R}{m}\cdot P(1-q)\right]\cdot C^o(m)\}$$

(VII.18)

There is no easy closed procedure which solves Equation (VII.18). Let us first express the fact, that at the optimal value of m_R (denote it by $m_R^o(m)$) we have:

$$C^{o}(m) = C_D \cdot m + (C_R - C_D) m_R^{o} + (1-P)^{m_R^{o}-1} \cdot [1-P + \frac{m_R^{o}}{m} \cdot P(1-q)] C^{o}(m) \quad . \qquad \text{(VII.19)}$$

We now define:

$$C(m_R;m) = \frac{C_D \cdot m + (C_R - C_D) m_R}{1-(1-P)^{m_R-1} \cdot [1-P + \frac{m_R}{m} \cdot P(1-q)]} \quad . \qquad \text{(VII.20)}$$

From Eq. (VII.19) and the definition of the function $C(m_R;m)$ we notice that at the solution point $m_R = m_R^{o}(m)$ we must have

$$C(m_R^{o}(m);m) = C^{o}(m) = \min_{1 \le m_R \le m} C(m_R;m) \quad .$$

The number $m_R^{o}(m)$ is the value of m_R (less than or equal to m) which minimizes $C(m_R;m)$. To find $m_R^{o}(m)$ let us first find the minimum of $C(m_R;m)$ when m_R is not constrained to the integers. After differentiating $C(m_R;m)$ and setting the derivative equal to zero, we obtain the following equation:

$$1 - r_c = (1-P)^{m_R-1} [a_0(m) + a_1(m) m_R + a_2(m) m_R^2] \qquad \text{(VII.21)}$$

where $a_0(m)$, $a_1(m)$ and $a_2(m)$ are constants (depending on m) given by:

$$a_0(m) = 1-P-r_c(1-Pq) + r_c \cdot m(1-P) \cdot \ln(\frac{1}{1-P}) \quad . \qquad \text{(VII.21a)}$$

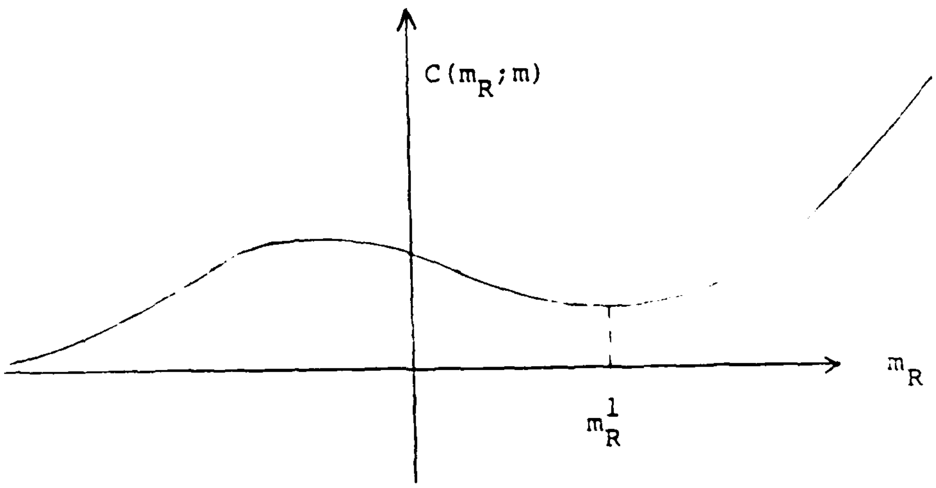
$$a_1(m) = [(1-r_c)(1-P) + r_c P(1-q)] \cdot \ln(\frac{1}{1-P}) \quad . \qquad \text{(VII.21b)}$$

$$a_2(m) = \frac{(1-r_c)P(1-q)}{m} \cdot \ln(\frac{1}{1-P}) \quad . \qquad \text{(VII.21c)}$$

Eq.(VII.21) can't be solved analytically. We must resort to some numerical procedure to solve it. For small values of m this is not, however, necessary. One can check directly where the minimum of the function $C(m_R;m)$ occurs by computing this function for all integer values of m_R between 1 and m. Notice that as $m_R \to \infty$, $C(m_R;m)$ tends to be linear with positive slope C_R-C_D. As $m_R \to -\infty$, we have $C(m_R;m) \to 0$. It can also be shown that

$$\left.\frac{dC(m_R;m)}{dm_R}\right|_{m_R=0} < 0$$

By taking the second derivative we also find that the first derivative (taken as a function of m_R) has exactly two roots (which means that Eq. (VII.21) has exactly two different solutions). The function $C(m_R;m)$ therefore has the following tentative form:



Let m_R^1 be the solution to Eq. (VII.21). From the above figure we learn that m_R^1 is always positive. Four cases are possible:

(a) $\underline{m_R^1 < 1.}$ In this case the solution to the actual problem is

$$m_R^o(m) = 1$$

that is, only one real missile should be used (together with m-1 decoys).

(b) $\underline{m_R^1 \geq m.}$ In this case

$$m_R^o(m) = m$$

i.e., all m missiles should be real.

(c) $\underline{1 \leq m_R^1 < m \text{ and } C([m_R^1];m) \leq C([m_R^1]+1;m).}$ In this case:

$$m_R^o(m) = [m_R^1] .$$

(d) $\underline{1 \leq m_R^1 < m \text{ and } C([m_R^1];m) > C([m_R^1]+1;m).}$ Then

$$m_R^o(m) = [m_R^1] + 1 .$$

So far we have solved the problem of finding the optimal mixture of real missiles and decoys when the total number of missiles to be launched was given (equal to m). For each value of m we can find $C^o(m)$ given by:

$$C^o(m) = C(m_R^o(m);m)$$

The solution to the original problem is thus:

$$C^o = \min_{m \geq 1} C^o(m) . \qquad \text{(VII.22)}$$

Since we cannot express $m_R^O(m)$ and $C(m_R^O(m);m)$ analytically, the optimization problem given in Eq. (VII.22) cannot be solved by any method other than a numerical search method. In the example given in the next section we actually carry out that search procedure.

To carry out a meaningful analysis of the effectiveness of decoys, we should compare the minimum achievable cost when decoys are actually available for use with the minimum achievable cost when only real missiles are available. If we denote by C^R this last quantity, we see that we are interested in the ratio

$$\frac{C^O}{C^R} \quad (P,\ q,\ r_c \text{ are } \underline{\text{given}})$$

The value of C^R can be found in much the same way as C^O. Since C^R is the minimal expected cost of destruction when only real missiles are available, the functional equation which C^R should satisfy is

$$C^R = \operatorname*{Min}_{m \geq 1} \{m \cdot C_R + (1-P)^{m-1} \cdot (1-Pq) \cdot C^R\} . \tag{VII.23}$$

We solve this problem using arguments similar to those we have given in solving the problem of finding $C^O(m)$. First we define the function $C^R(m)$ by:

$$C^R(m) = \frac{mC_R}{1-(1-P)^{m-1} \cdot (1-Pq)} . \tag{VII.24}$$

Let m^{OR} be the solution to Eq. (VII.23). It is immediately seen that

$$C^R = C^R(m^{OR}) .$$

Thus we have to find the minima of $C^R(m)$ on the (positive) integers. To do that, we first find the minima of $C^R(m)$ when m is not restricted to be an integer. By direct differentiation we find that the minima should satisfy the equation:

$$(1-Pq)\ [1 + \ln(\frac{1}{1-P})\cdot m\ ]\cdot(1-P)^{m-1} = 1 . \qquad \text{(VII.25)}$$

It can be shown by calculus that Eq. (VII.25) has only one positive root (call it m^1). If $m^1 < 1$, then $m^o = 1$, and if $m^1 > 1$, then $m^{OR} = [m^1]$ or $m^{OR} = [m^1]+1$, depending on whether $C^R([m_R^1])$ is less than or greater than $C^R([m_R^1]+1)$.

Eq. (VII.25) can be solved either by a search method or by an iterative method as we show next. This iterative method is based on the well-known Banach Fixed Point Theorem. This theorem, when applied to real functions, defined on the real line, states that if f(x) is a function with the contraction property, i.e., if

$$|f(x_2) - f(x_1)| < K\cdot|x_2 - x_1|$$

for any pair x_1,x_2, for some constant $K < 1$, then the equation:

$$f(x) = x$$

can be solved by the successive approximation method. In other words, if we take an arbitrary point x_0 as an initial approximation, and then define

$$x_n = f(x_{n-1}) \quad \text{for} \quad n = 1,2,3, \ldots$$

then it is guaranteed that

$$\lim_{n\to\infty} x_n = \bar{x} \quad \text{where} \quad f(\bar{x}) = \bar{x} .$$

Now to apply this method to our problem, notice that Eq. (VII.25) can be rewritten as:

$$m = 1 - \frac{\ln(1-Pq)}{\ln(1-P)} - \frac{\ln[1-\ln(1-P)\cdot m]}{\ln(1-P)}$$

so that, if we define

$$f(m) = 1 - \frac{\ln(1-Pq)}{\ln(1-P)} - \frac{\ln[1-\ln(1-P)\cdot m]}{\ln(1-P)}$$

the problem becomes:

$$m = f(m) .$$

To show that the method of successive approximation works, we need only to show that f(m) has the contraction property. So let m_1, m_2 be any two points on the real line (assume $m_1 > m_2$). We have:

$$|f(m_1) - f(m_2)| = \left|\frac{\ln[1-\ln(1-P)\cdot m_1]}{\ln(1-P} - \frac{\ln[1-\ln(1-P)\cdot m_2]}{\ln(1-P)}\right|$$

$$= \frac{1}{|\ln(1-P)|}\cdot\left|\ln\left(\frac{1-\ln(1-P)\cdot m_1}{1-\ln(1-P).m_2}\right)\right| = \frac{1}{|\ln(1-P)|}\left|\ln\left(1 - \frac{\ln(1-P)(m_1-m_2)}{1-\ln(1-P)\cdot m_2}\right)\right|$$

using now the inequality $\log(1+x) \leq x$ we find that the last expression is less than or equal to:

$$\frac{1}{|\ln(1-P)|}\cdot\frac{|\ln(1-P)|\,(m_1-m_2)}{1+|\ln(1-P)|\cdot m_2} < \frac{1}{1+|\ln(1-P)|}\cdot(m_1-m_2)$$

so we have proven

$$|f(m_1)-f(m_2)| < K\cdot(m_1-m_2)$$

where

$$K = \frac{1}{1+|\ln(1-P)|} < 1,$$

so that the contraction property indeed holds. In practice, we can stop the iterative calculation of the sequence m_n, given by $f(m_{n-1}) = m_n$, whenever it seems apparent that the sequence is "trapped" between two consecutive integers. This is implied by the fact that the solution of the actual problem is either the greatest integer smaller than the solution of $f(m) = m$ or the smallest integer greater than it.

Now let m^{OR} be the solution to the optimization problem presented by Eq. (VII.23). Then, from Eq. (VII.24) we have:

$$C^R = C^R(m^{OR}) = \frac{m^{OR}\cdot C_R}{1-(1-P)^{m^{OR}-1}(1-Pq)}. \tag{VII.26}$$

Let also m^O and m_R^O represent the optimal solution to the problem of optimal real-decoy mixture (m^O--total number of missiles (reals and decoys) to be launched, m_R^O--number of reals in the mixture). From Eq. (VII.20) we get:

$$C(m_R^o;m^o) = C^o = \frac{C_D \cdot m^o + (C_R - C_D) m_R^o}{1-(1-P)^{m_R^o - 1}\left[1-P + \frac{m_R^o}{m^o} P(1-q)\right]}, \qquad \text{(VII.27)}$$

so we have, from (VII.26) and (VII.27):

$$\frac{C^o}{C^R} = \frac{r_c \cdot m^o + (1-r_c) m_R^o}{m^{oR}} \times \frac{1-(1-P)^{m^{oR}-1} \cdot (1-Pq)}{1-(1-P)^{m_R^o - 1} \cdot \left[1-P + \frac{m_R^o}{m^o} \cdot P(1-q)\right]} .$$

2. Numerical Example

Table VII.1 presents the results of some numerical cases of the optimal mixture problem. In carrying out the calculations we have followed the following steps, for any given combination of P, q, and r_c:

(1) First, we calculated the optimal number m^{oR} of real missiles to be launched simultaneously (where use of decoys is not allowed) in order to minimize expected cost of destruction. The calculation is carried out by solving Eq. (VII.25), using the successive approximation method.* The values of m^{oR} are presented in the fourth column of Table VII.1.

(2) The optimal cost (C^R) of destruction (with real missiles only) is presented in the fifth column of the table. The cost is measured in units of C_R, the cost of a single real missile. It was calculated using Eq. (VII.26).

(3) For m = 1,2,3,... , the solution to Eq. (VII.18) was found, using the method described in Section C. For each m, we have calculated $m_R^o(m)$, the optimal number

*The successive approximation algorithm for Eq. (VII.25) was programmed on a TI-59 handheld calculator.

Table VII-1

Numerical Examples of the Optimal Real/Decoy Mixture Problem

Input Data			Output				
1	2	3	4	5	6	7	8
Prob. of Kill (given survival)	Prob. of survival	Decoy-to real cost ratio	Optimal No. of real missiles (decoys not available)	Optimal (scaled) cost of destruction (real missiles only)	Optimal Mixture (Reals, Decoys)	Optimal (scaled) cost of Dest.	$\frac{C^O}{C^R}$
(P)	(q)	(r_c)	(m^{OR})	(C^R/C_R)		(C^O/C_R)	
0.2	0.1	0.1	4	13.3	(2,3)	7.6	0.57
0.2	0.1	0.5	4	13.3	(4,0)	13.3	1
0.2	0.5	0.1	3	7.1	(2,1)	6.8	0.95
0.2	0.5	0.5	3	7.1	(3,0)	7.1	1
0.5	0.1	0.1	2	3.8	(1,3)	3.35	0.88
0.5	0.1	0.5	2	3.8	(2,0)	3.8	1
0.5	0.5	0.1	2	3.2	(1,2)	2.9	0.9
0.5	0.5	0.5	2	3.2	(2,0)	3.2	1

of missiles that should be real, if a total of m missiles are to be launched simultaneously (hence, $m-m_R^o(m)$ is the optimal number of decoys). Using Eq. (VII.20) we also calculated the value of $C(m_R^o(m);m)/C_R$ for each m. This is the optimal cost of destruction (again, in terms of C_R), for any given m.

(4) By keeping track of the variation of $C(m_R^o(m);m)$ as m increases, the value $m = m^o$ for which $C(m_R^o(m);m)$ attains its minimum was detected. The optimal mixture thus contains $m_R^o(m^o)$ real missiles and $m^o-m_R^o(m^o)$ decoys. It is presented in column 6 of Table VII.1 in the form (m_R^o, m_D^o) where $m_D^o = m^o-m_R^o$.

(5) The optimal cost of destruction (C^o), measured in terms of C_R, is shown in the seventh column of Table VII.1.

(6) The ratio of optimal cost with decoys (calculated in (5) above) to the optimal cost without decoys (calculated in step (2)) is presented in column 8. This is the natural measure of the effectiveness of decoys in saturating the defense system.

We have included in the tables the solution to the optimal mixture problem for all combinations of the following values of the parameters:

$P = 0.2, 0.5$

$q = 0.1, 0.5$

$r_c = 0.1, 0.5$

<u>Analysis of the results</u>. It is apparent from the results shown in Table VII-1 that the gain obtained by introducing decoys is more significant when the parameters P, q, r_c get smaller. Notice for instance, that for all cases for which $r_c = 0.5$ the optimal mixture is a pure "real" mixture, whereas

for all cases for which $r_c \approx 0.1$ the optimal mixture contains at least one decoy. This reflects the sensitivity of the optimal mixture to the cost of decoys. Notice also that the savings which result from using decoys (optimally!) becomes more significant as q, the probability of survival, gets smaller. If $P = 0.2$, $r_c = 0.1$ and $q \approx 0.1$, for instance, the decoys reduce the cost of destruction to only 57 percent of the cost which would be incurred using only real missiles. If q is raised to 0.5, the savings is much smaller, the optimal cost being only 0.95 of the cost incurred without decoys.

APPENDIX

FORMULAE FOR COMPUTING VALUE AND OPTIMAL STRATEGIES IN A 2 × 2 MATRIX GAME

The following elementary formulae are used repeatedly in Chapter IV and Chapter VI. Their derivation can be found in Owen [4].

Let A be a 2 × 2 matrix of a zero-sum game:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

We assume that player 1 selects the row, player 2 selects the column. The payoff a_{ij} is paid by player 1 to player 2, if they choose actions i and j, respectively. We denote by V the value of the game. The optimal strategies are completely determined by the probabilities π^1, π^2, where:

π^1 = Probability that player 1 selects the first row.

π^2 = Probability that player 2 selects the first column.

The formulae we use are:

$$V = \frac{a_{11} \cdot a_{22} - a_{12} \cdot a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} \tag{1}$$

$$\pi^1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} \tag{2}$$

$$\pi^2 = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}} \tag{3}$$

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